Queen Mary, University of London

MAS 412 (MTHM N64) Relativity and Gravitation

2 May 10:00, 2002, Time Allowed: 3 Hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.
Calculators are NOT permitted in this examination.

You are reminded of the following.

PHYSICAL CONSTANTS

Gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
Speed of light	c	3×10^8 m s^{-1}
1 kpc		3×10^{19} m

NOTATION

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature $(+ - - -)$ is used.

Partial derivatives are denoted by ",".

Covariant derivatives are denoted by ";".

USEFUL FORMULAS, which you may use without proof.

i) In an inertial reference system in Cartesian coordinates the metric interval is

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

(Minkowski space-time).

ii) The Christoffel symbols are

$$\Gamma^i_{km} = \frac{1}{2} g^{in} \left(g_{kn,m} + g_{mn,k} - g_{km,n} \right),$$

iii) The covariant derivatives of vectors $A^i = g^{ik}A_k$ and $A_i = g_{ik}A^k$ are

$$A^i_{;k} = A^i_{,k} + \Gamma^i_{km}A^m,$$

$$A_{i;k} = A_{i,k} - \Gamma^m_{ik}A_m.$$

iv) The geodesic equations are

$$\frac{du^i}{ds} + \Gamma^i_{kn}u^k u^n = 0,$$

where $u^i = dx^i/ds$ is 4-velocity.

v) The Riemann tensor is

$$R^i_{klm} = g^{in}R_{nklm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}.$$

vi) The symmetry properties of the Riemann tensor are

$$R_{iklm} = -R_{kilm} = -R_{ikml},$$

$$R_{iklm} = R_{lmik}.$$

vii) The equation for the geodesic deviation is

$$\frac{D^2\eta^i}{Ds^2} = R^i_{klm}u^k u^l \eta^m,$$

where η^i is the 4-vector joining points on two infinitesimally close geodesics, and u^k is the 4-velocity along the geodesic.

viii) The Bianchi identity is

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0;$$

Exam continued

ix) The Ricci tensor and the scalar curvature are

$$R_{ik} = g^{lm} R_{limk} = R^m_{imk},$$

$$R = g^{il} g^{km} R_{iklm} = g^{ik} R_{ik} = R^i_i.$$

x) The Einstein equations are

$$R_{ik} = \frac{8\pi G}{c^4} \left(T_{ik} - \frac{1}{2} g_{ik} T \right).$$

xi) The spherically symmetric Schwarzschild metric interval is

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{(1 - \frac{r_g}{r})} - r^2 \left(\sin^2\theta d\phi^2 + d\theta^2 \right),$$

where $r_g = 2GM/c^2$ is the gravitational radius of the central body with mass M.
xii) The Kerr metric, describing a rotating black hole is

$$ds^2 = (1 - \frac{r_g r}{\rho^2}) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2} \sin^2\theta) \sin^2\theta d\phi^2$$

$$+ \frac{2 r_g r a}{\rho^2} \sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2 \cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the specific angular momentum of the black hole.
xiii) A weak gravitational wave is a small perturbation of the galilean metric

$$g_{\alpha\beta} = -\delta_{\alpha\beta} + h_{\alpha\beta}.$$

xiv) The physical distance between two close test particles in the field of a weak gravitational wave is

$$L = L_0 (1 + \frac{1}{2} h_{\alpha\beta} n^\alpha n^\beta),$$

where $\eta^\alpha = L_0 n^\alpha$ is the unperturbed separation vector between these particles.
xv) The quadrupole formula for the generation of gravitational waves is

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R} \frac{d^2 D_{\alpha\beta}}{dt^2},$$

where

$$D_{\alpha\beta} = \int (3 x_\alpha x_\beta - r^2 \delta_{\alpha\beta}) dM$$

is the quadrupole tensor.

Exam continued

SECTION A: Each question carries 8 marks.You should attempt ALL questions.

A1.

In a uniformly rotating system of coordinates (rotation about $z^{'}$-axis) $x^{'}$, $y^{'}$, $z^{'}$ the interval ds has the following form:

$$ds^2 = g_{ik}dx^i dx^k = g^{'}_{ik}dx^{'i}dx^{'k} =$$

$$[c^2 - \Omega^2(x^{'2} + y^{'2})]dt^{'2} - dx^{'2} - dy^{'2} - dz^{'2} + 2\Omega y^{'}dx^{'}dt^{'} - 2\Omega x^{'}dy^{'}dt^{'}.$$

Show, that by proper choice of coordinates it is possible to reduce this interval to Minkowski form.

A2.

Give the definition of a mixed tensor of the second rank in terms of the transformation of curvilinear coordinates. Show by straightforward differentiation that if A_i^k is a tensor then dA_i^k is not a tensor.

Write down the covariant derivative of the mixed tensor of the second rank in terms of the Christoffel symbols.

A3.

By straightforward calculation show that for any arbitrary covariant vector A_i the difference $A_{i;k;l} - A_{i;l;k}$ is given by the Riemann tensor R^m_{ikl} and the Riemann tensor satisfies

$$A_{i;k;l} - A_{i;l;k} = A_m R^m_{ikl}.$$

A4.

Two test particles move along two close radial geodesics in an unknown gravitational field. At some moment of time the separation between these particle is $l = 1cm$ in the radial direction and the relative acceleration of the particles in the radial direction is $a = 10\ cm/sec^2$. Using the equation for geodesic deviation, evaluate R^1_{001} in the frame of reference co-moving with one of the test particles.

What can you say about a gravitational field and a frame of reference, if i) all components of the Riemann tensor everywhere are equal to zero, while the metric interval is different from the Minkowski one, ii) at some event at least one component of the Riemann tensor is not equal to zero, while the metric interval at this event is the Minkowski one?

Exam continued

A5.

Using the Einstein equations, show that the stress-energy of matter can be written in terms of R^i_k and R as follows

$$T^i_k = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right),$$

where δ^i_k is the unit diagonal four-tensor. What can you say about the nature of gravitational fields for which $R_{ik} = 0$, while R_{ikln} is not equal to zero?

A6.

Using the Bianchi identity, prove that

$$R^l_{m;l} = \frac{1}{2}R_{,m},$$

and then show that the energy-momentum tensor of matter T^i_k satisfies the conservation law

$$T^k_{i;k} = 0.$$

A7.

Using the Kerr metric, find the location of the outer event horizon, r_{hor}. Show that $r_{hor} < r_g$, where r_g is the radius of the event horizon of the non-rotating black hole of the same mass.

Exam continued

SECTION B: Each question carries 22 marks. You may attempt all questions but only marks for the best 2 questions will be counted.

B1.

Find the form of the Schwarzschild metric after the following transformation of coordinates

$$c\tau = ct + \int \frac{(r_g r)^{1/2} dr}{(r - r_g)}, \quad R = ct + \int \frac{r^{3/2} dr}{r_g^{1/2}(r - r_g)}$$

and discuss its behavior at $r = r_g$. Express r in terms of R and τ and show that the final metric is non-stationary.

Using the equation $ds = 0$ for $\theta,\ \phi = const$, consider the propagation of radial light signals in the Schwarzschild space-time. Show that

$$\frac{d(c\tau)}{dR} = \pm\sqrt{\frac{r_g}{r}}.$$

By analysis of the positions of the lines $r = const$ with respect to the light cones on the diagram $(c\tau, R)$, give the definition of the event horizon.

B2.

A test particle moves along a circular orbit with radius r in the equatorial plane ($\theta = \pi/2$) of a spherically symmetric gravitational field given by the Schwarzschild metric. Write down the geodesic equations for such a particle. Calculate all Christoffel symbols which you need to find the angular velocity $\Omega = d\phi/dt$ of the particle. Show that $\Omega = \Omega_N$, where

$$\Omega_N = \sqrt{\frac{GM}{r^3}}$$

is the corresponding angular velocity in Newtonian theory for the circular orbit of the same radius.

Solve the same problem in the Kerr metric. Consider two particles, one of which rotates in the same direction as the Kerr black hole, and the other rotates in the opposite direction. Show that the difference between their angular velocities is

$$\Delta\Omega = \frac{2c\Omega_N^2 a}{c^2 - a^2\Omega_N^2},$$

where a is the angular momentum parameter in the Kerr metric. Which of these two particles moves faster?

Exam continued

B3.

A star of mass m moves along a circular orbit of radius $r \gg R_g$ around a massive black hole of mass $M \gg m$. Using the quadrupole formula for the generation of gravitational waves, show that to order of magnitude the amplitude of a gravitational wave at a distance $R \gg r$ from the black hole is

$$h \approx \frac{r_g R_g}{rR} \approx \frac{r_g}{R}\left(\frac{2\pi R_g}{cP}\right)^{2/3},$$

where $r_g = 2Gm/c^2$ is the gravitational radius of the star and P is the orbital period of the star around the black hole.

A detector of gravitational waves consists of two test particles, separated by the distance $L = 10^6\ km$. Assume that the orbit of the above binary lies in the (x^2, x^3)-plane, the centre of the detector is situated on the x^1-axis, and the separation vector between the two test particles is parallel to the x^3-axis. Evaluate the amplitude, $\delta L/L$, and frequency, Ω, of variations of the distance between the two test particles, if $R = 10^4\ pc$ (the distance to the center of our Galaxy), $r = 10^8\ km$, $m = M_\odot$ and $M = 10^6 M_\odot$, where $M_\odot$ is the solar mass. (Take into account that the gravitational radius of the Sun is equal to $3km$.)

END OF EXAMINATION

A.Polnarev