Solutions, 2002

A1. Unseen

In a uniformly rotating system of coordinates (rotation about $z^{'}$-axis) $x^{'}$, $y^{'}$, $z^{'}$ the interval ds has the following form:

$$ds^2 = g_{ik}dx^i dx^k = g^{'}_{ik}dx^{'i}dx^{'k} =$$
$$[c^2 - \Omega^2(x^{'2} + y^{'2})]dt^{'2} - dx^{'2} - dy^{'2} - dz^{'2} + 2\Omega y^{'}dx^{'}dt^{'} - 2\Omega x^{'}dy^{'}dt^{'}.$$

Show, that by proper choice of coordinates it is possible to reduce this interval to Minkowski form.

The transformation should be back rotation around $z^{'}$-axis,
so

$$t^{'} = t,\ z^{'} = z,$$
$$x^{'} = x\cos\phi + y\sin\phi,\ y^{'} = xsin\phi - y\cos\phi,$$

where $\phi = -\Omega t$.

[1 Mark]

Then

$$x^{'2} + y^{'2} = (x\cos\phi + y\sin\phi)^2 + (x\sin\phi - y\cos\phi)^2 = x^2 + y^2,$$

[0.5 Mark]

$$dt^{'} = dt,\ dz^{'} = dz,$$
$$dx^{'} = dx\cos\phi + dy\sin\phi + (x\sin\phi - y\cos\phi)\Omega dt,$$
$$dy^{'} = dx\sin\phi - dy\cos\phi - (x\cos\phi + y\sin\phi)\Omega dt.$$

[0.5 Mark]

Then

$$dx^{'2} + dy^{'2} = (dx\cos\phi + dy\sin\phi) - (x\sin\phi + y\cos\phi)\Omega dt)^2 +$$
$$(-dx\sin\phi - dy\cos\phi - (x\cos\phi - y\sin\phi)\Omega dt)^2 =$$
$$dx^2 + dy^2 + (x^2 + y^2)(\Omega dt)^2 - 2\Omega dt(ydx - xdy),$$

[2 Marks]

$$y^{'}dx^{'} - x^{'}dy^{'} = (-xsin\phi - y\cos\phi)(dx\cos\phi - dy\sin\phi - (x\sin\phi + y\cos\phi)\Omega dt) -$$
$$(x\cos\phi - y\sin\phi)(-dx\sin\phi - dy\cos\phi - (x\cos\phi - y\sin\phi)\Omega dt) =$$
$$= (x^2 + y^2)\Omega dt - (ydx - xdy).$$

[2 Marks]

Thus

$$ds^2 = [c^2 - \Omega^2(x^2 + y^2)]dt^2 - dx^2 - dy^2 - dz^2 -$$
$$\Omega^2(x^2 + y^2)dt^2 + 2\Omega dt(ydx - xdy) + 2\Omega^2(x^2 + y^2)]dt^2 + 2\Omega dt(xdy - ydx)$$
$$= c^2dt^2 - dx^2 - dy^2 - dz^2.$$

[2 Marks]

A2. Bookwork

Give the definition of a mixed tensor of the second rank in terms of the transformation of curvilinear coordinates. Show by straightforward differentiation that if A_i^k is a tensor then dA_i^k is not a tensor.
Write down the covariant derivative of the mixed tensor of the second rank in terms of the Christoffel symbols.

A mixed tensor A_k^i transforms as the product of two vectors A^k and A_i, thus

$$A_k^i = \frac{\partial x^i}{\partial x'^n}\frac{\partial x'^m}{\partial x^k}A'^n_m,$$

[2 Marks]

This is the only definition of mixed tensor of the second rank.

[1 Mark]

$$dA_i^k = d[\frac{\partial x^i}{\partial x'^n}\frac{\partial x'^m}{\partial x^k}]A'^n_m + \frac{\partial x^i}{\partial x'^n}\frac{\partial x'^m}{\partial x^k}dA'^n_m$$
$$= [\frac{\partial^2 x^i}{\partial x'^n \partial x'^l}\frac{\partial x'^m}{\partial x^k} + \frac{\partial x^i}{\partial x'^n}\frac{\partial^2 x'^m}{\partial x^k \partial x'^l}]A'^n_m dx'^l + \frac{\partial x^i}{\partial x'^n}\frac{\partial x'^m}{\partial x^k}dA'^n_m.$$

[2 Marks]

The first term in general case is not zero, hence dA_i^k is not a tensor.

[1 Mark]

Using again the fact, that A_k^i transforms as the product of two vectors A^k and A_i, we have

$$A^i_{k,l} = A^i_{k,l} + \Gamma^i_{lm}A^m_k - \Gamma^m_{lk}A^i_m.$$

[2 Marks]

A3. Bookwork

By straightforward calculation show that for any arbitrary covariant vector A_i the difference $A_{i;k;l} - A_{i;l;k}$ is given by the Riemann tensor R^m_{ikl} and the Riemann tensor satisfies

$$A_{i;k;l} - A_{i;l;k} = A_m R^m_{ikl}.$$

$$A_{i;k;l} - A_{i;l;k} = (A_{i,k} - \Gamma^m_{ik}A_m)_{;l} - (A_{i,l} - \Gamma^m_{il}A_m)_{;k} =$$

[2 Marks]

$$(A_{i,k} - \Gamma^m_{ik}A_m)_{,l} - \Gamma^n_{il}(A_{n,k} - \Gamma^m_{nk}A_m)$$
$$-\Gamma^n_{kl}(A_{i,n} - \Gamma^m_{in}A_m) - (A_{i,l} - \Gamma^m_{il}A_m)_{,k} + \Gamma^n_{ik}(A_{n,l} - \Gamma^m_{nl}A_m) + \Gamma^n_{lk}(A_{i,n} - \Gamma^m_{in}A_m) =$$

[2 Marks]

$$A_{i,k,l} - A_{i,l,k} - A_{m,l}\Gamma^m_{ik} - A_{n,k}\Gamma^n_{il} - A_{i,n}\Gamma^n_{kl} + A_{m,k}\Gamma^m_{il} + A_{n,l}\Gamma^n_{ik} + A_{i,n}\Gamma^n_{kl} +$$

$$A_m[-\Gamma^m_{ik,l} + \Gamma^n_{il}\Gamma^m_{nk} + \Gamma^n_{kl}\Gamma^m_{in} +$$

$$\Gamma^m_{il,k} - \Gamma^n_{ik}\Gamma^m_{nl} - \Gamma^n_{lk}\Gamma^m_{in}] =$$

[2 Marks]

$$A_m(\Gamma^m_{il,k} - \Gamma^m_{ik,l} + \Gamma^n_{il}\Gamma^m_{nk} - \Gamma^n_{ik}\Gamma^m_{nl}) = A_m R^m_{ikl}$$

[2 Marks]

A4. Unseen

Two test particles move along two close radial geodesics in an unknown gravitational field. At some moment of time the separation between these particle is $l = 1cm$ in the radial direction and the relative acceleration of the particles in the radial direction is $a = 10\ cm/sec^2$. Using the equation for geodesic deviation, evaluate R^1_{001} in the frame of reference co-moving with one of the test particles.

What can you say about a gravitational field and a frame of reference, if i) all components of the Riemann tensor everywhere are equal to zero, while the metric interval is different from the Minkowski one, ii) at some event at least one component of the Riemann tensor is not equal to zero, while the metric interval at this event is the Minkowski one?

$$a = \frac{D^2\eta^1}{D\tau^2} = \frac{c^2 d^2\eta^1}{ds^2}$$

where

$$d\tau = \frac{ds}{c}$$

is the proper time. Hence, from the equation for the geodesic deviation:

$$a = R^1_{kl1} u^k u^l l c^2$$

From the fact that we chose a co-moving frame

$$u^k = \delta^k_0$$

[2 Marks]

$$a = R^1_{001} l c^2$$

and

$$R^1_{001} = \frac{a}{lc^2} = \frac{10 \cdot cms^2}{s^2 \cdot 1cm \cdot 3 \cdot 10^{10} cm^2} \approx \frac{1}{3 \cdot 10^9} cm^{-2} \approx 3 \cdot 10^{-10} \frac{1}{cm^2}$$

[2 Marks]

i) $R_{iklm} = 0$ means the space-time is flat. The fact that metric interval is different from the Minkowski one means that a curvilinear coordinate system is used.

[2 Marks]

ii)$R_{iklm} \neq 0$ means that the space-time is curved. The fact that the metric interval at this event is the Minkowski one means, that a local galilean frame of reference is used.

[2 Marks]

A5. Bookwork

Using the Einstein equations, show that the stress-energy of matter can be written in terms of R^i_k and R as follows

$$T^i_k = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right),$$

where δ^i_k is the unit diagonal four-tensor. What can you say about the nature of gravitational fields for which $R_{ik} = 0$, while R_{ikln} is not equal to zero?

Contracting with g^{ik}, we have the Einstein equations in mixed form

$$R^i_k = \frac{8\pi G}{c^4}(T^i_k - \frac{1}{2}\delta^i_k T).$$

[2 Marks]

$$R = g^{ik}R_{ik} = \frac{8\pi G}{c^4}\left(g^{ik}T_{ik} - \frac{1}{2}g^{ik}g_{ik}T\right) = \frac{8\pi G}{c^4}\left(T^i_i - \frac{1}{2}\delta^i_i T\right) = \frac{8\pi G}{c^4}\left(T - \frac{1}{2}4\right) = -\frac{8\pi G}{c^4}T.$$

Thus

$$T = -\frac{c^4}{8\pi G}R.$$

[2 Marks]

Thus

$$T_{ik} = \frac{c^4}{8\pi G}\left(R_{ik} - \frac{1}{2}g_{ik}R\right),$$

then in mixed form we have

$$T^i_k = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right).$$

[2 Marks]

This situation corresponds to gravitational fields (for example, gravitational waves), when the space-time is curved, but matter is absent (empty space-time).

[2 Marks]

A6. Bookwork

Using the Bianchi identity, prove that

$$R^l_{m;l} = \frac{1}{2}R_{,m},$$

and then show that the energy-momentum tensor of matter T^i_k satisfies the conservation law

$$T^k_{i;k} = 0.$$

Contracting the Bianchi identity on the pairs of indices ik and ln we have

$$g^{ik}(R^l_{ikl;m} + R^l_{imk;l} + R^l_{ilm;k}) = g^{ik}g^{lp}(R_{pikl;m} + R_{pimk;l} + R_{pilm;k}) =$$
$$g^{ik}g^{lp}(-R_{ipkl;m} - R_{ipmk;l} - R_{iplm;k}) =$$

[2 Marks]

By symmetry properties of the Riemann tensor

$$g^{ik}g^{lp}(R_{pikl;m} + R_{pimk;l} + R_{pilm;k}) =$$

$$g^{ik}g^{lp}(-R_{klip;m} - R_{mkip;l} - R_{lmip;k}) = g^{ik}g^{lp}(-R_{klip;m} + R_{kmip;l} - R_{lmip;k}) =$$

[2 Marks]

By the definition of the Ricci tensor

$$-g^{lp}(R_{lp;m} - R_{mp;l} + g^{ik}R_{mi;k} = -R, m + R^l_{m;l} + R^k_{m;k} = -R_{,m} + 2R^l_{m;l} = 0,$$

thus

$$R^l_{m;l} = \frac{1}{2}R_{,m}.$$

[2 Marks]

From A4 we have

$$T^i_{k;i} = \frac{c^4}{8\pi G}\left(R^i_{k;i} - \frac{1}{2}\delta^i_k R, i\right) = \frac{c^4}{8\pi G}\left(\frac{1}{2}R_{,k} - \frac{1}{2}R, k\right) = 0.$$

[2 Marks]

A7. Bookwork

Using the Kerr metric, find the location of the outer event horizon, r_{hor}. Show that $r_{hor} < r_g$, where r_g is the radius of the event horizon of the non-rotating black hole of the same mass.

In the Kerr metric, as well as in the case of a non-rotating black hole, the location of the horizon corresponds to $g_{11} \longrightarrow \infty$.

[2 marks]

This means $\Delta = 0$:

[1 mark]

$$r^2 - r_g r + a^2 = 0,$$

$$r_\pm = \frac{r_g \pm \sqrt{r_g^2 - a^2}}{2} = \frac{r_g}{2}\left(1 \pm \sqrt{1 - \frac{a^2}{r_g^2}}\right).$$

[2 marks]

1) There are two horizons instead of one as in the case $a = 0$.

[1 mark]

2) The radius of the outer horizon is always less than in the Schwarzschild case,

$$r_+ = \frac{r_g + \sqrt{r_g^2 - a^2}}{2} = \frac{r_g}{2}\left(1 + \sqrt{1 - \frac{a^2}{r_g^2}}\right) < r_g .$$

[2 marks]

SECTION B: Each question carries 22 marks. You may attempt all questions but only marks for the best 2 questions will be counted.

B1. Bookwork

Find the form of the Schwarzschild metric after the following transformation of coordinates

$$c\tau = ct + \int \frac{(r_g r)^{1/2} dr}{(r - r_g)}, \quad R = ct + \int \frac{r^{3/2} dr}{r_g^{1/2}(r - r_g)}$$

and discuss its behavior at $r = r_g$*. Express* r *in terms of* R *and* τ *and show that the final metric is non-stationary.*

Using the equation $ds = 0$ *for* θ*,* $\phi = const$*, consider the propagation of radial light signals in the Schwarzschild space-time. Show that*

$$\frac{d(c\tau)}{dR} = \pm\sqrt{\frac{r_g}{r}}.$$

By analysis of the positions of the lines $r = const$ *with respect to the light cones on the diagram* $(c\tau, R)$*, give the definition of the event horizon.*

$$cd\tau = cdt + \frac{(r_g r)^{1/2} dr}{(r - r_g)}, \quad dR = cdt + \frac{r^{3/2} dr}{r_g^{1/2}(r - r_g)}.$$

[1 mark]

$$dR - cd\tau = \frac{dr}{r - r_g}\left(\frac{r^{3/2}}{r_g^{1/2}} - r_g^{1/2} r^{1/2}\right) = \frac{dr r_g^{1/2}}{r^{1/2}},$$

hence

$$dr = \frac{r^{1/2}}{r_g^{1/2}}\left(dR - cd\tau\right).$$

[2 mark]

$$cdt = cd\tau - \frac{(r_g r)^{1/2}}{(r - r_g)}\frac{r_g^{1/2}}{r^{1/2}}\left(dR - cd\tau\right) = \frac{crd\tau - r_g dR}{r - r_g}.$$

[1 mark]

$$ds^2 = \left(1 - \frac{r_g}{r}\right)\frac{1}{(r - r_g)^2}(crd\tau - r_g dR)^2 - \frac{r_g}{r}\frac{(dR - cd\tau)^2}{1 - \frac{r_g}{r}}$$

$$-r^2(\sin^2\theta d\phi^2 + d\theta^2) = \frac{1}{r(r - r_g}[(crd\tau - r_g dR)^2 - r_g r(dR - cd\tau)^2]$$

$$-r^2(\sin^2\theta d\phi^2 + d\theta^2) = \frac{1}{r(r - r_g}[c^2 d\tau^2(r^2 - r_g r)$$

$$-dR^2(r_g r - r_g^2) + 2dRcd\tau(-2rr_g + 2rr_g)] - r^2(\sin^2\theta d\phi^2 + d\theta^2)$$

$$= \frac{1}{r}(rc^2d\tau^2 - r_g dR^2] - r^2(\sin^2\theta d\phi^2 + d\theta^2)r^2$$
$$= c^2d\tau^2 - \frac{r_g}{r}dR^2 - r^2(\sin^2\theta d\phi^2 + d\theta^2).$$

[4 marks]

When $r = r_g$
$$ds^2 = c^2d\tau^2 - dR^2 - r_g^2(\sin^2\theta d\phi^2 + d\theta^2),$$
which is not singular, in contrast to the original form.

[1 mark]

$$r^{1/2}dr = r_g^{1/2}(dR - cd\tau),$$
$$\frac{2}{3}r^{3/2} = r_g^{1/2}(R - c\tau) + C,$$
we can choose $C = 0$, so
$$r = \left[\frac{3}{2}(R - c\tau)\right]^{2/3} r_g^{1/3}.$$

[3 marks]

$$ds^2 = c^2d\tau^2 - \left[\frac{3}{2}(R - c\tau)\right]^{-2/3} dR^2 - r_g^{2/3}\left[\frac{3}{2}(R - c\tau)\right]^{4/3}(\sin^2\theta d\phi^2 + d\theta^2).$$

g_{11}, g_{22} and g_{33} are functions of time τ, which means non-stationarity.

[2 marks]

For radial light propagation $d\theta = d\phi = 0$, so from $ds = 0$ we have
$$\frac{cd\tau}{R} = \pm\sqrt{\frac{r_g}{r}}.$$

[2 marks]

$r = const$ means $R - c\tau = const$, so

$c\tau$

$r = 0$, $\quad r < r_g \quad r = r_g \quad r > r_g$

[4 marks]

The event horizon is null surface $r = r_g$ and no signal can escape to infinity from the inside of this surface.

[2 marks]

B2. Unseen

A test particle moves along a circular orbit with radius r *in the equatorial plane* $(\theta = \pi/2)$ *of a spherically symmetric gravitational field given by the Schwarzschild metric. Write down the geodesic equations for such a particle. Calculate all Christoffel symbols which you need to find the angular velocity* $\Omega = d\phi/dt$ *of the particle. Show that* $\Omega = \Omega_N$*, where*

$$\Omega_N = \sqrt{\frac{GM}{r^3}}$$

is the corresponding angular velocity in Newtonian theory for the circular orbit of the same radius.
Solve the same problem in the Kerr metric. Consider two particles, one of which rotates in the same direction as the Kerr black hole, and the other rotates in the opposite direction. Show that the difference between their angular velocities is

$$\Delta\Omega = \frac{2c\Omega_N^2 a}{c^2 - a^2\Omega_N^2},$$

where a *is the angular momentum parameter in the Kerr metric. Which of these two particles moves faster?*

As particle moves along circular orbit in equatorial plane $u^1 = dx^1/ds = dr/ds = 0$, $d^2r/ds^2 = 0$, $u^2 = dx^2/ds = d\theta/ds = 0$,$d^2r/ds^2 = 0$, $\theta = \pi/2$.

[1 mark]

Thus we have

$$\begin{aligned}
\frac{du^0}{ds} + \Gamma^0_{00}u^0u^0 + 2\Gamma^0_{03}u^0u^3 + \Gamma^0_{33}u^3u^3 &= 0,\\
\Gamma^1_{00}u^0u^0 + 2\Gamma^1_{03}u^0u^3 + \Gamma^1_{33}u^3u^3 &= 0,\\
\Gamma^2_{00}u^0u^0 + 2\Gamma^2_{03}u^0u^3 + \Gamma^2_{33}u^3u^3 &= 0,\\
\frac{du^3}{ds} + \Gamma^3_{00}u^0u^0 + 2\Gamma^3_{03}u^0u^3 + \Gamma^3_{33}u^3u^3 &= 0.
\end{aligned}$$

[2 marks]

From the second equation we have

$$\Gamma^1_{00} + 2\Gamma^1_{03}\frac{\Omega}{c} + \Gamma^1_{33}\frac{\Omega^2}{c^2} = 0.$$

[2 marks]

So we should calculate Γ^1_{00}, Γ^1_{03} and Γ^1_{33}.

[1 mark]

For the Schwarzschild metric

$$\Gamma^1_{00} = \frac{1}{2}g^{11}\left(g_{01,0} + g_{01,0} - g_{00,1}\right) = \frac{1}{2}(1-\frac{r_g}{r})\frac{d}{dr}(1-\frac{r_g}{r}) =$$

$$= \frac{r_g}{2r^2}(1-\frac{r_g}{r}),$$

$$\Gamma^1_{03} = \frac{1}{2}g^{11}\left(g_{01,3} + g_{01,3} - g_{03,1}\right) = 0,$$

[1 mark]

$$\Gamma^1_{33} = \frac{1}{2}g^{11}\left(g_{31,3} + g_{31,3} - g_{33,1}\right) = -\frac{1}{2}(1-\frac{r_g}{r})\frac{d}{dr}(r^2) = -(r-r_g),$$

[2 marks]

So we obtain

$$\frac{r_g}{2r^2}(1-\frac{r_g}{r}) - (r-r_g)\left(\frac{d\phi}{cdt}\right)^2 = 0,$$

$$\frac{\Omega^2}{c^2} = \frac{r_g}{2r^3},$$

$$\Omega^2 = \frac{c^2 r_g}{2r^3} = \frac{c^2 \cdot GM}{c^2 \cdot 2r^3} = \frac{GM}{r^3}.$$

$$\Omega = \Omega_N = \sqrt{\frac{GM}{r^3}},$$

the same as in Newtonian case.

[3 marks]

In the Kerr metric

$$\Gamma^1_{00} = \frac{1}{2}g^{11}(g_{01,0} + g_{01,0} - g_{00,1}) = -\frac{1}{2}g^{11}g_{00,1}$$

$$= -\frac{1}{2}g^{11}\left(1-\frac{r_g r}{\rho^2}\right)_{,1} = -\frac{1}{2}g^{11}\left(1-\frac{r_g}{r}\right)_{,1} = -g^{11}\frac{r_g}{2r^2},$$

$$\Gamma^1_{03} = \frac{1}{2}g^{11}(g_{01,3}+g_{31,0}-g_{03,1}) = -\frac{1}{2}g^{11}\left(\frac{r_g r a}{\rho^2}\sin^2\theta\right)_{,1} = -\frac{1}{2}g^{11}\left(\frac{r_g a}{r}\right)_{,1} = \frac{1}{2}g^{11}\left(\frac{r_g a}{r^2}\right),$$

[2 marks]

$$\Gamma^1_{33} = \frac{1}{2}g^{11}(g_{31,3} + g_{31,3} - g_{33,1}) = \frac{1}{2}g^{11}\left[\left(r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta\right)\sin^2\theta\right]_{,1}$$

$$= -\frac{1}{2}g^{11}\left(r^2 + a^2 + \frac{r_g a^2}{r}\right)_{,1} = \frac{1}{2}g^{11}\left(2r - \frac{r_g a^2}{r^2}\right)$$

$$= g^{11} r\left(1 - \frac{r_g a^2}{2r^3}\right),$$

[2 marks]

and we have

$$-\frac{r_g}{2r^2}g^{11} + g^{11}\frac{r_g a}{r^2}\frac{\Omega}{c} + g^{11}r\left(1 - \frac{r_g a^2}{2r^3}\right)\frac{\Omega^2}{c^2} = 0,$$

$$-\frac{1}{c^2}\Omega_N^2 + \frac{2a}{c^3}\Omega_N^2\Omega + \left(1 - \frac{a^2\Omega_N^2}{c^2}\right)\frac{\Omega^2}{c^2} = 0,$$

[2 marks]

$$\Omega^2 - \Omega^2\left(1 - \frac{2a\Omega}{c} + \frac{a^2\Omega^2}{c^2}\right) = 0,$$

$$\Omega^2 - \Omega_N^2\left(1 - \frac{a\Omega}{c}\right)^2 = 0,$$

$$\left[\Omega - \Omega_N\left(1 - \frac{a\Omega}{c}\right)\right]\left[\Omega + \Omega_N\left(1 - \frac{a\Omega}{c}\right)\right] = 0,$$

$$\Omega_+\left(1 + \frac{a\Omega_N}{c}\right) = \Omega_N, \quad \Omega_+ = \frac{\Omega_N c}{c + a\Omega_N}, \quad \Omega_- = -\frac{\Omega_N c}{c - a\Omega_N},$$

$$\Delta\Omega = |\Omega_-| - |\Omega_+| = \frac{2\Omega_N^2 ac}{c^2 - a^2\Omega_N^2},$$

$$|\Omega_-| > |\Omega_+|,$$

[3 marks]

which means retrograde particle moves faster.

[1 mark]

B3. Unseen

A star of mass m moves along a circular orbit of radius $r \gg R_g$ around a massive black hole of mass $M \gg m$. Using the quadrupole formula for the generation of gravitational waves, show that to order of magnitude the amplitude of a gravitational wave at a distance $R \gg r$ from the black hole is

$$h \approx \frac{r_g R_g}{rR} \approx \frac{r_g}{R}\left(\frac{2\pi R_g}{cP}\right)^{2/3},$$

where $r_g = 2Gm/c^2$ is the gravitational radius of the star and P is the orbital period of the star around the black hole.

A detector of gravitational waves consists of two test particles, separated by the distance $L = 10^6$ km. Assume that the orbit of the above binary lies in the (x^2, x^3)-plane, the centre of the detector is situated on the x^1-axis, and the separation vector between the two test particles is parallel to the x^3-axis. Evaluate the amplitude, $\delta L/L$, and frequency, Ω, of variations of the distance between the two test particles, if $R = 10^4$ pc (the distance to the center of our Galaxy), $r = 10^8$ km, $m = M_\odot$ and $M = 10^6 M_\odot$, where $M_\odot$ is the solar mass. (Take into account that the gravitational radius of the Sun is equal to 3km.)

To order of magnitude, the quadrupole is

$$D \sim mr^2,$$

[1 mark]

Hence

$$\ddot{D} \sim mr^2\omega^2,$$

[1 mark]

which means

$$h \sim \frac{Gmr^2\omega^2}{c^4R} = \frac{Gmr^2GM}{c^4Rr^3} = \frac{r_gR_g}{Rr}.$$

[2 marks]

Expressing r in terms of ω, we obtain

$$r^3 = \frac{GM}{\omega^2}, \quad r = \left(\frac{GM}{\omega^2}\right)^{1/3} \sim \left(\frac{R_gc^2}{\omega^2}\right)^{1/3},$$

[1 mark]

and

$$h \sim \frac{r_gR_g}{R}\left(\frac{\omega^2}{R_gc^2}\right)^{1/3} = \frac{r_g}{R}\left(\frac{R_g\omega}{c}\right)^{2/3}.$$

[3 marks]

$x^1 = x$, $x^2 = y$, $x^3 = z$.

[1 mark]

$$\frac{\delta L}{L} = \frac{1}{2}h_{\alpha\beta}n^\alpha n^\beta = \frac{1}{2}h_{33},$$

[1 mark]

$$h_{33} = -\frac{2G}{3c^4R}\ddot{D}_{33} = -\frac{2G}{3c^4R}m\ddot{(3z^2 - r^2)} = -\frac{2G}{3c^4R}m(\ddot{z^2}) = -\frac{r_g}{c^2R}(2\dot{z\dot{z}}) = -\frac{2r_g}{c^2R}\left(\ddot{z}z + \dot{z}^2\right).$$

[3 marks]

$$z = r\cos\omega t,$$

$$\dot{z} = -\omega r\sin\omega t,$$

$$\ddot{z} = -\omega^2 r \cos\omega t,$$

$$\ddot{z}z + \dot{z}^2 = \omega^2 r^2 \left(-\cos^2\omega t + \sin^2\omega t\right) = -\omega^2 r^2 \cos 2\omega t,$$

$$\frac{\delta L}{L} = \frac{r_g r^2 \omega^2}{c^2 R} \cos 2\omega t,$$

[3 marks]

$$\delta L = \frac{r_g r^2 \omega^2 L}{c^2 R} \cos 2\omega t,$$

[2 marks]

$$\omega^2 = \frac{GM}{r^3},$$

$$\delta L = \frac{1}{2} \frac{r_g R_g L}{rR} \cos 2\omega t.$$

$$|\delta L| = \frac{3\ km \cdot 3 \cdot 10^6\ km \cdot 10^6\ km}{2 \cdot 10^8\ km \cdot 10^4 \cdot 3 \cdot 10^{13}\ km} = 1.5 \cdot 10^{-13}\ km = 1.5 \cdot 10^{-8} cm.$$

[2 marks]

Frequency

$$\Omega = 2\omega = 2\sqrt{\frac{GM}{r^3}} = \frac{c\sqrt{2}}{r}\sqrt{\frac{R_g}{r}} = \frac{3 \cdot 10^5\ km\sqrt{2}}{sec \cdot 10^8\ km}\sqrt{\frac{3 \cdot 10^6\ km}{10^8\ km}} = 3\sqrt{6} \cdot 10^{5-1-8} \cdot sec^{-1} \approx 7 \cdot 10^{-4}\ Hz.$$

[2 marks]