Queen Mary
University of London

M.Sc. EXAMINATION

MAS 412 (MTHM N64) Relativity and Gravitation

Monday, 16 May 2005
14:30-17:30

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

You are reminded of the following.

PHYSICAL CONSTANTS

Gravitational constant	G	$= 6.7 \times 10^{-11}$ N m^2 kg^{-2}
Speed of light	c	$= 3 \times 10^{8}$ m s^{-1}
1 kpc		$= 3 \times 10^{19}$ m

NOTATION

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature $(+ - - -)$ is used.

Partial derivatives are denoted by ",".

Covariant derivatives are denoted by ";".

USEFUL FORMULAS, which you may use without proof.

Minkowski metric:

$$ds^2 = \eta_{ik} dx^i dx^k = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Christoffel symbol:

$$\Gamma^i_{km} = \frac{1}{2} g^{in} \left(g_{kn,m} + g_{mn,k} - g_{km,n} \right),$$

Covariant derivatives:

$$A^i_{;k} = A^i_{,k} + \Gamma^i_{km} A^m, \quad A_{i;k} = A_{i,k} - \Gamma^m_{ik} A_m.$$

Geodesic equation:

$$\frac{du^i}{ds} + \Gamma^i_{kn} u^k u^n = 0, \quad \text{where } \ u^i = dx^i/ds \ \text{ is 4-velocity.}$$

Riemann tensor:

$$A^i_{;k;l} - A^i_{;l;k} = -A^m R^i_{mkl}, \quad \text{where } \ R^i_{klm} = g^{in} R_{nklm},$$

$$R_{iklm} = \frac{1}{2} \left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l} \right) + g_{np} \left(\Gamma^n_{kl} \Gamma^p_{im} - \Gamma^n_{km} \Gamma^p_{il} \right).$$

Symmetry properties of the Riemann tensor:

$$R_{iklm} = -R_{kilm} = -R_{ikml}, \quad R_{iklm} = R_{lmik}.$$

Geodesic deviation equation:

$$\frac{D^2 \eta^i}{ds^2} = R^i_{klm} u^k u^l \eta^m,$$

where η^i is the 4-vector joining points on two infinitesimally close geodesics, and u^k is the 4-velocity along the geodesic.

Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

Ricci tensor:

$$R_{ik} = g^{lm} R_{limk} = R^m_{imk}.$$

Scalar curvature:

$$R = g^{il} g^{km} R_{iklm} = g^{ik} R_{ik} = R^i_i.$$

Einstein equations:

$$R^i_k - \frac{1}{2} \delta^i_k R = \frac{8\pi G}{c^4} T^i_k.$$

Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right).$$

Gravitational radius:

$$r_g = 2GM/c^2 = 3(M/M_\odot) \text{ km}, \text{ where } M_\odot \text{ is the mass of Sun.}$$

Kerr metric:

$$ds^2 = (1 - \frac{r_g r}{\rho^2})dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2$$

$$+\frac{2r_g r a}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the specific angular momentum.

Hamilton-Jacobi equation:

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} - m^2c^2 = 0,$$

where four-momentum $p_i = -\frac{\partial S}{\partial x^i}$ and $p_0 = E$ (energy), $p_3 = L$ (angular momentum).

Quadrupole formula for gravitational waves:

$$h_{\alpha\beta} = -\frac{2G}{3c^4R}\frac{d^2D_{\alpha\beta}}{dt^2}, \text{ where } D_{\alpha\beta} = \int(3x_\alpha x_\beta - r^2\delta_{\alpha\beta})dM \text{ is the quadrupole tensor.}$$

SECTION A

Each question carries 8 marks. You should attempt all questions.

1. (a) [**5 Marks**] A spacecraft moves around a planet of mass m and radius r along a circular orbit of radius $R = 2r$. Ignoring the transverse Doppler effect, evaluate the redshift z of the radio signal emitted by a probe left on the surface of the planet and received by the spacecraft.

(b) [**3 Marks**] Another spacecraft of height h moves very far from any gravitating bodies with acceleration a. Show that the redshift of a photon emitted at the bottom of the rocket and detected at its top is $z = ah/c^2$. [Hint: First solve the problem as for part (a) for radii r and $R = r + h$, where $h \ll r$; then apply the equivalence principle.]

2. (a) **[3 Marks]** Give the definition of the mixed tensor of the second rank in terms of the transformation of curvilinear coordinates (you can assume that a mixed tensor of the second rank is transformed as a product of covariant and contrvariant vectors).

(b) **[5 Marks]** In the non-rotating system of Cartesian coordinates $(x,\ y,\ z)$

$$A^i_k = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using coordinate transformation from Cartesian to the uniformly rotating cylindrical coordinates $(r,\ \theta,\ \phi)$

$$x = r\cos(\theta + \Omega t),\quad y = r\sin(\theta + \Omega t),\quad z = Z,$$

show that in the latter coordinates

$$A'^1_0 = -\frac{r\Omega}{2c}\sin 2(\theta + \Omega t).$$

3. (a) **[4 Marks]** Using the formulae for the Cristoffel symbol and covariant derivatives given in the rubric or otherwise, show that covariant derivatives of contrvariant metric tensor are equal to zero, $g^{ik}{}_{;n} = 0$.

(b) **[4 Marks]** Using the Einstein Equations show that in empty space-time

$$A^n{}_{;n;l} = A^n{}_{;l;n}$$

for an arbitrary covariant vector A_i.

4. (a) **[3 Marks]** A gravitational field is described by the interval

$$ds^2 = t^2 \eta_{ik} dx^i dx^k.$$

Show that all non vanishing components of the Cristophel symbol can be represented in the following form

$$\Gamma^i_{nk} = \frac{1}{t}\gamma^i_{nk},\quad \text{where}\quad \gamma^i_{nk} = \delta^0_n \delta^i_k + \delta^0_k \delta^i_n - \delta^i_0 \eta_{kn}.$$

(b) **[5 Marks]** Show that the scalar curvature R of the above field is equal to zero.

5. (a) **[6 Marks]** Using the Einstein equations, the Bianchi identity and the symmetry properties of the Riemann tensor, show that covariant divergence of the stress-energy tensor is equal to zero.

(b) [**2 Marks**] Take the stress-energy tensor in the form

$$T^i_k = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix},$$

where ε is energy density and p is pressure (if $p > 0$) or tension (if $p < 0$). Using the Einstein equations, evaluate the scalar curvature in terms of ε and p.

6. (a) [**3 Marks**] The four-velocity and the four-momentum of a particle of mass m in a gravitational field are defined as

$$u^i = \frac{dx^i}{ds}, \quad p^i = mcu^i.$$

Show that $u_i u^i = 1$ and $p_i p^i = m^2c^2$.

(b) [**5 Marks**] Show that in a static gravitational field with metric interval

$$ds^2 = g_{00}(dx^0)^2 + g_{\alpha\beta}dx^\alpha dx^\beta,$$

the energy of the particle, $E = mc^2u_0$, is given by

$$E = \frac{mc^2\sqrt{g_{00}}}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where

$$v = \frac{c\sqrt{-g_{\alpha\beta}dx^\alpha dx^\beta}}{\sqrt{g_{00}}dx^0}.$$

7. (a) [**4 Marks**] Using the Kerr metric, find the location of the event horizon, r_{hor}, and the limit of stationarity, r_{st}. Compare these results with the case of a non-rotating black hole.

(b) [**4 Marks**] Show that the circle defined by $r = r_{hor}$ and $\theta = \pi/2$, is the world line of a photon moving around the rotating black hole with angular velocity

$$\Omega_{hor} = \frac{a}{r_g r_{hor}}.$$

[Next section overleaf.]

SECTION B

Each question carries 22 marks. Only marks for the best TWO questions will be counted.

1. (a) [**4 Marks**] Consider the motion of a particle in the equatorial plane ($\theta = \frac{\pi}{2}$) of the spherically symmetric Schwarzshild gravitational field . Given that the solution of the Hamilton-Jacobi equation can be written in the following form

$$S = -Et + L\phi + S_r(r),$$

where the constants $E = mc^2 u_0$ and $L = mcu_3$ are the energy and angular momentum of the particle, find a differential equation for S_r.

(b) [**7 Marks**] Show that

$$E\left(1 - \frac{r_g}{r}\right)^{-1}\frac{dr}{dt} = c\sqrt{E^2 - U_{\text{eff}}^2},$$

where U_{eff} is the "effective potential energy" is given by

$$U_{\text{eff}}(r) = mc^2\sqrt{\left(1 - \frac{r_g}{r}\right)\left(1 + \frac{L^2}{m^2c^2r^2}\right)}.$$

(c) [**11 Marks**] Explain why the condition $E > U_{\text{eff}}(r)$ determines the admissible range of the motion. Solve the simultaneous equations $U_{\text{eff}}(r) = E$ and $U'_{\text{eff}}(r) = 0$ to show that the radius of the stable circular orbit with angular momentum L is

$$r = \frac{L^2}{m^2c^2r_g}\left[1 + \sqrt{1 - \frac{3m^2c^2r_g^2}{L^2}}\right].$$

Evaluate the radius of the innermost stable circular orbit.

2. (a) [**5 Marks**] Using the equation $ds = 0$ with $\theta,\ \phi = \text{const}$, consider the propagation of radial light signals in the Schwarzschild space-time. Consider a photon emitted outward from $r = r_0$ at time $t = 0$. Show that the world-line of the photon is given by

$$ct = r - r_0 + r_g \ln\frac{r - r_g}{r_0 - r_g}.$$

(b) [**10 Marks**] A particle moves along a radial geodesic in the Schwarzschild metric. Using the expression for ds and an appropriate component of geodesic equation, show that if the particle starts to fall freely from infinity, then

$$r(\tau) = \left[r^{3/2}(\tau_0) - \frac{3}{2}cr_g^{1/2}(\tau - \tau_0)\right]^{2/3},$$

where τ is the proper time ($ds = cd\tau$).

(c) [**7 Marks**] A free-falling observer moves radially with zero velocity at infinity in the gravitational field of Schwarzschild black hole. When it passes the radius $r_0 \gg r_g$ he starts to send outward radio-pulses with constant rate. The very small time interval between two subsequent pulses measured by clocks of the observer is equal to $\Delta\tau \ll r_g/c < r/c$. The second observer resting very far from the black hole receives signals sent by the first observer. Show that the time interval between the $(n+1)^{th}$ and the n^{th} pulse depends on n according to

$$\Delta t_n = \frac{\Delta\tau}{1-\sqrt{\frac{r_g}{r_n}}},$$

where r_n is the radius at which the n^{th} pulse is emitted.

3. (a) [**10 Marks**] A weak gravitational wave is a small perturbation of the Minkovski metric, $g_{ik} = \eta_{ik} + h_{ik}$. Show that $g^{ik} = \eta^{ik} - \eta^{in}\eta^{km}h_{nk}$. Use a linear coordinate transformation

$$x'^i = x^i + \xi^i,$$

where ξ^i are small functions of x^i, to impose on h_{ik} the following four supplementary conditions

$$\eta^{km}h_{mi,k} - \frac{1}{2}\delta_i^k\eta^{nm}h_{nm,k} = 0.$$

Show that after such transformation the Ricci tensor is reduced to

$$R_{ik} = -\frac{1}{2}\eta^{lm}\frac{\partial^2 h_{ik}}{\partial x^l \partial x^m}.$$

(b) [**5 Marks**]Consider a ring of test particles initially at rest in the (y, z)-plane, perturbed by a plane monochromatic gravitational wave propagating in x-direction with frequency ω and amplitude h_0. Explain what is meant by "+" and "×" polarizations. Sketch the shape of the ring at $x = 0$ and at times $t = 0$, $\frac{\pi}{2\omega}$, $\frac{\pi}{\omega}$, $\frac{3\pi}{2\omega}$ and $\frac{2\pi}{\omega}$ for two different polarizations of the gravitational wave: (i) $h_+ = h_0 \sin\omega(t-x/c)$, $h_\times = 0$; and (ii) $h_+ = 0$, $h_\times = h_0 \sin\omega(t-x/c)$.

(c) [**7 Marks**] Two bodies of equal mass $m_1 = m_2 = m$, attracting each other according to Newton's law, move in circular orbits around their common centre of mass with orbital period P. Using the quadrupole formula for the generation of gravitational waves show that in order of magnitude

$$h \sim \frac{r_g}{R}\left(\frac{r_g}{cP}\right)^{2/3},$$

where R is the distance to the system and $r_g = \frac{2Gm}{c^2}$ is the gravitational radius.

[End of examination paper.]