Queen Mary
University of London

M.Sc. EXAMINATION

MAS 412 (MTHM N64) Relativity and Gravitation

Monday, 16 May 2005
14:30-17:30

Solutions

SECTION A

Each question carries 8 marks.

1. (a) [**5 Marks**] A spacecraft moves around a planet of mass m and radius r along a circular orbit of radius $R = 2r$. Ignoring the transverse Doppler effect, evaluate the redshift z of the radio signal emitted by a probe left on the surface of the planet and received by the spacecraft.

(b) [**3 Marks**] Another spacecraft of height h moves very far from any gravitating bodies with acceleration a. Show that the redshift of a photon emitted at the bottom of the rocket and detected at its top is $z = ah/c^2$. [Hint: First solve the problem as for part (a) for radii r and $R = r + h$, where $h \ll r$; then apply the equivalence principle.]

A1(a)(*seen similar*)

•[**2 Marks**]

From conservation of energy, neglecting transverse Doppler effect, we have

$$h\nu_{ob} - \frac{Gm}{R}\frac{h\nu_{ob}}{c^2} = h\nu_{em} - \frac{Gm}{R}\frac{h\nu_{em}}{c^2}.$$

•[**1 Mark**]

Thus

$$\frac{\nu_{ob}}{\nu_{em}} = \frac{1 - \frac{Gm}{rc^2}}{1 - \frac{Gm}{Rc^2}}.$$

•[**2 Marks**]

Taking into account that in Newtonian limit $Gm/rc^2 \ll 1$, we have

$$\frac{\nu_{ob}}{\nu_{em}} \approx 1 - \frac{Gm}{rc^2}\left(1 - \frac{r}{R}\right) = 1 - \frac{Gm}{2rc^2},$$

then

$$z = \frac{\nu_{em} - \nu_{ob}}{\nu_{em}} = 1 - \frac{\nu_{ob}}{\nu_{em}} = \frac{GM}{rc^2}(1 - \frac{r}{R}) = \frac{Gm}{2rc^2}.$$

A1(b)(*seen similar*)

•[**2 Marks**]

If $R = r + h$ and $h \ll r$

$$z = \frac{GM}{rc^2}(1 - \frac{r}{r+h}) \approx \frac{GM}{rc^2}(1 - (1 - \frac{h}{r})) = \frac{GMh}{r^2c^2} = \frac{gh}{c^2},$$

where g is free fall acceleration at the surface of gravitating body.

•[**1 Mark**]

According to the equivalence principle

$$z = \frac{ah}{c^2}.$$

2. (a) **[3 Marks]** Give the definition of the mixed tensor of the second rank in terms of the transformation of curvilinear coordinates (you can assume that a mixed tensor of the second rank is transformed as a product of covariant and contrvariant vectors).

(b) **[5 Marks]** In the non-rotating system of Cartesian coordinates $(x,\ y,\ z)$

$$A^i_k = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using coordinate transformation from Cartesian to the uniformly rotating cylindrical coordinates $(r,\ \theta,\ \phi)$

$$x = r\cos(\theta + \Omega t), \quad y = r\sin(\theta + \Omega t), \quad z = Z,$$

show that in the latter coordinates

$$A'^1_0 = -\frac{r\Omega}{2c}\sin 2(\theta + \Omega t).$$

A2(a)(*seen similar*)

•**[1 Mark]**

A contravariant vector A^i transforms as

$$A^i = \frac{\partial x^i}{\partial x'^n} A'^n,$$

and this is the only definition of the contravariant vector.

•**[1 Mark]**

A covariant vector B_i transforms as

$$B_i = \frac{\partial x'^n}{\partial x^i} B'_n,$$

and this is the only definition of the covariant vector.

•**[1 Mark]**

A mixed tensor C^i_k transforms as product of A^i and B_k:

$$C^i_k = A^i B_k = \frac{\partial x^i}{\partial x'^n} A'^n \frac{\partial x'^m}{\partial x^k} B'_m = \frac{\partial x^i}{\partial x'^n}\frac{\partial x'^m}{\partial x^k} C'^n_m.$$

and this is the definition of the mixed tensor of the second rank.

A2(b)(*unseen*)

•**[2 Marks]**

$$A'^1_0 = \frac{\partial x'^1}{\partial x^n}\frac{\partial x^m}{\partial x^0} A^n_m = \frac{\partial r}{\partial x}\frac{\partial x}{c\partial T},$$

 [This question continues overleaf . . .]

•**[1 Mark]**
taking into account that

$$r = \sqrt{x^2 + y^2}, \quad \frac{\partial r}{\partial x} = \frac{x}{r} = \cos[2(\theta + \Omega t)],$$

and

$$\frac{\partial x}{c\partial T} = \frac{\partial x}{c\partial t} = -\frac{\Omega y}{c},$$

•**[2 Marks]**
we have thus

$$A_0^{\prime 1} = -\frac{\Omega y}{c}\cos(\theta + \Omega t) = -\frac{r\Omega}{2c}\sin 2(\theta + \Omega t).$$

3. (a) **[4 Marks]** Using the formulae for the Cristoffel symbol and covariant derivatives given in the rubric or otherwise, show that covariant derivatives of contrvariant metric tensor are equal to zero, ${g^{ik}}_{;n} = 0$.

(b) **[4 Marks]** Using the Einstein Equations show that in empty space-time

$$A^n{}_{;n;l} = A^n{}_{;l;n}$$

for an arbitrary covariant vector A_i.

A3(a)(*seen similar*)

•**[1 Mark]**
The relation

$$DA^i = g^{ik} DA_k$$

is valid for covariant differential as for any vector.

•**[2 Marks]**
On other hand

$$DA^i = D(g^{ik} A_k) = g^{ik} DA_k + A_k Dg^{ik},$$

thus

$$g^{ik} DA_k = g^{ik} DA_k + A_k Dg^{ik}.$$

Taking into account that A^i is arbitrary vector we have

•**[1 Mark]**

$$Dg^{ik} = g^{ik}_{;n} dx^n = 0$$

for arbitrary dx^n. Hence

$$g^{ik}_{;n} = 0.$$

A3(b)(*unseen*)

•**[2 Marks]**
From the definition of the Riemann tensor

$$A^i_{;k;l} - A^i_{;l;k} = -A^m R^i_{mkl},$$

 [This question continues overleaf . . .]

by summation $i = k = n$ we have

$$A^n_{;n;l} - A^n_{;l;n} = -A^m R^n_{mnl} = -A^m R_{ml},$$

where R_{ml} is the Ricci tensor.

•**[2 Marks]**

According to the Einstein equations in empty space-time, i.e when stress-energy tensor vanishes, $T_{ik} = 0$, the Ricci tensor is also vanishes, hence

$$A^n_{;n;l} = A^n_{;l;n}.$$

4. (a) **[3 Marks]** A gravitational field is described by the interval

$$ds^2 = t^2 \eta_{ik} dx^i dx^k.$$

Show that all non vanishing components of the Cristophel symbol can be represented in the following form

$$\Gamma^i_{nk} = \frac{1}{t}\gamma^i_{nk}, \quad \text{where} \quad \gamma^i_{nk} = \delta^0_n \delta^i_k + \delta^0_k \delta^i_n - \delta^i_0 \eta_{kn}.$$

(b) **[5 Marks]** Show that the scalar curvature R of the above field is equal to zero.

A4(a)

•**[1 Mark]** (*unseen*)

$$g_{ik} = t^2 \eta_{ik}, \quad \text{hence} \quad g^{ik} = t^{-2}\eta^{ik},$$

•**[2 Marks]**

$$\Gamma^i_{km} = \frac{1}{2} g^{in}\left(g_{kn,m} + g_{mn,k} - g_{km,n}\right) = \frac{1}{2t^2}\eta^{im}(\delta^0_n \cdot 2t\eta_{km} + \delta^0_k \cdot 2t\eta_{nm} - \delta^0_m \cdot 2t\eta_{kn)} =$$

$$= \frac{1}{t}(\delta^0_n \delta^i_k + \delta^0_k \delta^i_n - \delta^0_m \eta^{im}\eta_{kn}) =$$

$$= \frac{1}{t}(\delta^0_n \delta^i_k + \delta^0_k \delta^i_n - \delta^i_0 \eta_{kn}) = \frac{1}{t}\gamma^i_{nk}.$$

A4(b)(*unseen*)

•**[2 Marks]**

$$R = g^{ik} R_{ik} = t^{-2}\eta^{ik}(-\delta^0_l \frac{1}{t^2}\gamma^l_{ik} + \delta^0_k \frac{1}{t^2}\gamma^l_{il} + \frac{1}{t^2}\gamma^l_{ik}\gamma^m_{lm} - \frac{1}{t^2}\gamma^m_{il}\gamma^l_{km}) =$$

$$= t^{-4}Q, \quad \text{where} \quad Q = -\eta^{ik}\gamma^0_{ik} + \gamma^l_{0l} + \eta^{ik}\gamma^l_{ik}\gamma^m_{lm} - \eta^{ik}\gamma^m_{il}\gamma^l_{km} =$$

•**[2 Marks]**

$$= -\eta^{ik}\gamma^0_{ik} + \gamma^l_{0l} + \eta^{ik}\gamma^l_{ik}(\delta^m_l\delta^0_m + \delta^m_m\delta^0_l - \delta m_0\eta_{lm}) -$$

$$-\eta^{ik}\gamma^m_{il}(\delta^l_k\delta^0_m + \delta^l_m\delta^0_k - \delta l_0\eta_{km}) =$$

$$= 2\eta^{ik}\gamma^0_{ik} + \gamma^l_{0l} = \delta^l_l\delta^0_0 + \delta^l_0\delta^0_l - \delta^l_0\eta_{0l} + 2\eta^{ik}(\delta^0_i\delta^0_k + \delta^0_i\delta^0_k - \delta^0_0\eta_{ik}) =$$

•**[1 Mark]**

$$4 + 1 - 1 + 2(2-4) = 0.$$

5. (a) **[5 Marks]** Using the Einstein equations, the Bianchi identity and the symmetry properties of the Riemann tensor, show that covariant divergence of the stress-energy tensor is equal to zero.

(b) **[2 Marks]** Take the stress-energy tensor in the form

$$T^i_k = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix},$$

where ε is energy density and p is pressure (if $p > 0$) or tension (if $p < 0$). Using the Einstein equations, evaluate the scalar curvature in terms of ε and p.

A5(a)*(seen similar)*

•**[1 Mark]**

Contracting the Bianchi identity on the pairs of indices ik and ln

$$g^{ik}(R^n_{ikn;m} + R^n_{imk;n} + R^n_{inm;k}) = 0,$$

and taking into account that covariant derivatives of the metric tensor are equal to zero we have

•**[2 Marks]**

$$[g^{ik}R^n_{ikn}]_{;m} + [g^{ik}R^n_{imk}]_{;n} + [g^{ik}R^n_{inm}]_{;k} = 0,$$

•**[2 Marks]**

using symmetry properties and the definition of the Ricci tensor and the scalar curvature we have

$$-[g^{ik}R^n_{ink}]_{;m} + [g^{ik}R^n_{imk}]_{;n} + [g^{ik}R^n_{inm}]_{;k} = 0,$$

$$-R_{,m} + R^n_{m;n} + R^k_{m;k} = 0$$

hence

$$R^n_{m;n} = \frac{1}{2}R_{,m}.$$

•**[1 Mark]**

Putting this into the Einstein Equations we have

$$T^i_{k;i} = \frac{c^4}{8\pi G}(R^i_{k;i} - \frac{1}{2}\delta^i_k R_{,i}) = 0.$$

A5(b)(*unseen*)

•**[2 Marks]**

Contracting the Einstein equations we have

$$R - \frac{1}{2}4R = \frac{8\pi G}{c^4}T, \text{ hence } R = -\frac{8\pi G}{c^4}T = -\frac{8\pi G}{c^4}(\varepsilon - 3p)$$

6. (a) **[3 Marks]** The four-velocity and the four-momentum of a particle of mass m in a gravitational field are defined as

$$u^i = \frac{dx^i}{ds}, \quad p^i = mcu^i.$$

Show that $u_i u^i = 1$ and $p_i p^i = m^2c^2$.

(b) **[5 Marks]** Show that in a static gravitational field with metric interval

$$ds^2 = g_{00}(dx^0)^2 + g_{\alpha\beta}dx^\alpha dx^\beta,$$

the energy of the particle, $E = mc^2 u_0$, is given by

$$E = \frac{mc^2\sqrt{g_{00}}}{\sqrt{1-\frac{v^2}{c^2}}},$$

where

$$v = \frac{c\sqrt{-g_{\alpha\beta}dx^\alpha dx^\beta}}{\sqrt{g_{00}}dx^0}.$$

A6(a)(*seen similar*)

•**[2 Marks]**

Starting from formula for the interval

$$ds^2 = g_{ik}dx^i dx^k,$$

and dividing both side of this equation by ds^2 we have

$$1 = g_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} = g_{ik}u^i u^k = u_i u^i =$$

•**[1 Mark]**

$$= \frac{1}{m^2c^2}p_i p^i, \text{ hence } p_i p^i = m^2c^2.$$

$u_i u^i = 1$ and $p_i p^i = m^2c^2$.

 [This question continues overleaf . . .]

A6(b)(*seen similar*)

•[**1 Mark**]

$$E = mc^2 u_0 = mc^2 g_{00} u^0 = mc^2 g_{00} \frac{dx^0}{ds} =$$

•[**2 Marks**]

$$= mc^2 g_{00} \frac{dx^0}{\sqrt{g_{00}(dx^0)^2 + g_{\alpha\beta} dx^\alpha dx^\beta}};$$

•[**2 Marks**]
introducing then the velocity

$$v = \frac{c\sqrt{-g_{\alpha\beta} dx^\alpha dx^\beta}}{\sqrt{g_{00}} dx^0},$$

we have

$$E = \frac{mc^2 \sqrt{g_{00}}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

7. (a) [**4 Marks**] Using the Kerr metric, find the location of the event horizon, r_{hor}, and the limit of stationarity, r_{st}. Compare these results with the case of a non-rotating black hole.

(b) [**4 Marks**] Show that the circle defined by $r = r_{hor}$ and $\theta = \pi/2$, is the world line of a photon moving around the rotating black hole with angular velocity

$$\Omega_{hor} = \frac{a}{r_g r_{hor}}.$$

A7(a)(*seen similar*)

•[**2 Marks**]

The location of horizon: $g_{11} = \infty$, hence

$$\Delta = r^2 - r_g r + a^2 = 0,$$

The larger solution is outer horizon:

$$r_{hor} = \frac{r_g}{2} + \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}.$$

$$r_{hor} \le \frac{r_g}{2} + \frac{r_g}{2} = r_g,$$

if $a = 0$, $r_{hor} = r_g$.

•[**2 Marks**]

The location of the limit of stationarity is the surface $g_{00} = 0$. For the Kerr metric $g_{00} = 0$ gives

$$1 - \frac{r_g r}{\rho^2} = 0,$$

thus

$$r^2 - r_g r + a^2 \cos^2\theta = 0,$$

$$r = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2\cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2\cos^2\theta}.$$

"+" corresponds to the outer surface $g_{00} = 0$ and we should take this solution.

A7(b)(*unseen*)

•**[1 Mark]**

As $\Delta = 0$ we have $r_{hor}^2 + a^2 = r_g r_{hor}$.

•**[3 Mars]**

For $d\phi = \Omega_{hor} dt$

$$\frac{ds^2}{c^2dt^2} = 1 - \frac{r_g}{r_{hor}} - (r_{hor}^2 + a^2 + \frac{r_g a^2}{r_{hor}})\Omega_{hor}^2 + \frac{2r_g a}{r_{hor}}\Omega_{hor} =$$

$$= 1 - \frac{r_g}{r_{hor}} - (r_g r_{hor} + \frac{r_g a^2}{r_{hor}})\Omega_{hor}^2 + \frac{2r_g a}{r_{hor}}\Omega_{hor} =$$

$$= 1 - \frac{r_g}{r_{hor}} - \frac{r_g}{r_{hor}}(r_{hor}^2 + a^2)\Omega_{hor}^2 + \frac{2r_g a}{r_{hor}}\Omega_{hor} =$$

$$= 1 - \frac{r_g}{r_{hor}} - r_g^2\Omega_{hor}^2 + \frac{2r_g a}{r_{hor}}\Omega_{hor} =$$

$$= 1 - \frac{r_g}{r_{hor}} - r_g^2(\frac{a}{r_g r_{hor}})^2 + \frac{2r_g a}{r_{hor}}\frac{a}{r_g r_{hor}} =$$

$$= 1 - \frac{r_g}{r_{hor}} - \frac{a^2}{r_{hor}^2} + 2\frac{a^2}{r_{hor}^2} = \frac{r_{hor}^2 - r_g r_{hor} + a^2}{r_{hor}^2} = \frac{\Delta}{r_{hor}^2} = 0.$$

Hence $r = r_{hor}$ and $\phi = \Omega_{hor} t$ correspond to the world-line of photon.

[Next section overleaf.]

SECTION B

Each question carries 22 marks.

1. (a) [**4 Marks**] Consider the motion of a particle in the equatorial plane ($\theta = \frac{\pi}{2}$) of the spherically symmetric Schwarzshild gravitational field. Given that the solution of the Hamilton-Jacobi equation can be written in the following form

$$S = -Et + L\phi + S_r(r),$$

where the constants $E = mc^2u_0$ and $L = mcu_3$ are the energy and angular momentum of the particle, find a differential equation for S_r.

(b) [**7 Marks**] Show that

$$E\left(1 - \frac{r_g}{r}\right)^{-1}\frac{dr}{dt} = c\sqrt{E^2 - U_{\text{eff}}^2},$$

where U_{eff} is the "effective potential energy" is given by

$$U_{\text{eff}}(r) = mc^2\sqrt{\left(1 - \frac{r_g}{r}\right)\left(1 + \frac{L^2}{m^2c^2r^2}\right)}.$$

(c) [**11 Marks**] Explain why the condition $E > U_{\text{eff}}(r)$ determines the admissible range of the motion. Solve the simultaneous equations $U_{\text{eff}}(r) = E$ and $U_{\text{eff}}^{'}(r) = 0$ to show that the radius of the stable circular orbit with angular momentum L is

$$r = \frac{L^2}{m^2c^2r_g}\left[1 + \sqrt{1 - \frac{3m^2c^2r_g^2}{L^2}}\right].$$

Evaluate the radius of the innermost stable circular orbit.

B1(a)(*seen similar*)

•[**2 Marks**]

Taking $\theta = \pi/2$ we can write down the Hamilton-Jacobi equation in the Schwarzschild metric as

$$\left(1 - \frac{r_g}{r}\right)^{-1}\left(\frac{\partial S}{c\partial t}\right)^2 - \left(1 - \frac{r_g}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2 - m^2c^2 = 0.$$

•[**2 Marks**]

Then putting $S = -Et + L\phi + S_r(r)$, we have

$$\left(1 - \frac{r_g}{r}\right)^{-1}\frac{E^2}{c^2} - \left(1 - \frac{r_g}{r}\right)\left(\frac{dS_r}{dr}\right)^2 - \frac{L^2}{r^2} - m^2c^2 = 0,$$

which is the usual differential equation for $S_r(r)$.

 [This question continues overleaf . . .]

B1(b)(*seen similar*)

•[**2 Marks**]

The radial component of the four-momentum can be found as

$$\frac{\partial S}{\partial r} = \frac{dS_r}{dr} = p_1 = g_{11}p^1 = g_{11}\frac{dr}{ds} = \sqrt{\frac{E^2}{c^2}\left(1-\frac{r_g}{r}\right)^{-2} - \left(m^2c^2 + \frac{L^2}{r^2}\right)\left(1-\frac{r_g}{r}\right)^{-1}} =$$

$$= \frac{1}{c}\left(1-\frac{r_g}{r}\right)^{-1}\sqrt{E^2 - m^2c^4\left(1+\frac{L^2}{m^2c^2r^2}\right)\left(1-\frac{r_g}{r}\right)}.$$

•[**1 Mark**]

On other hand

$$\frac{dt}{ds} = p^0 = g^{00}p_0 = g^{00}\left(\frac{\partial S}{\partial t}\right) = -g^{00}E.$$

•[**2 Marks**]

Thus

$$\frac{dr}{dt} = \frac{\frac{dr}{ds}}{\frac{dt}{ds}} = \frac{1}{c}\left(1-\frac{r_g}{r}\right)\sqrt{E^2 - U_{\text{eff}}^2}\frac{1}{E} = \frac{1}{c}\left(1-\frac{r_g}{r}\right)^{-1}\sqrt{E^2 - U_{\text{eff}}^2},$$

•[**2 Marks**]

where

$$U_{\text{eff}} = mc^2\sqrt{\left(1+\frac{L^2}{m^2c^2r^2}\right)\left(1-\frac{r_g}{r}\right)},$$

hence

$$E\left(1-\frac{r_g}{r}\right)^{-1}\frac{dr}{dt} = c\sqrt{E^2 - U_{\text{eff}}^2}.$$

B1(c)(*seen similar*)

•[**1 Mark**](*book work*)

For given radius U_{eff} is equal to the energy of a particle which has the turning point for this r, i.e. $dr/dt = 0$, thus the condition $E > U_{\text{eff}}$ determines the admissible range of the motion.

•[**1 Mark**](*book work*)

All circular orbits are determined by simultaneous solution of the equations

$$U_{\text{eff}} = E \text{ and } \frac{dU_{\text{eff}}}{dr} = 0.$$

•[**1 Mark**](*seen similar*)

From $dU_{\text{eff}}/dr = 0$ we have $dU_{\text{eff}}^2/du = 0$, where $u = 1/r$.

•[**2 Marks**]

 [This question continues overleaf ...]

Hence

$$-r_g\left(1+\frac{L^2u^2}{m^2c^2}\right)+(1-r_gu)\frac{2L^2u}{m^2c^2}=0,\quad \text{or}\quad r_gr^2+3r_g\left(\frac{L}{mc}\right)^2-2\left(\frac{L}{mc}\right)^2r=0.$$

•**[2 Marks]**
Solving this equation we have

$$r_\pm=\frac{L^2}{m^2c^2r_g}\pm\sqrt{\left(\frac{L^2}{m^2c^2r_g}\right)^2-\frac{3L^2}{m^2c^2}}=\frac{L^2}{m^2c^2r_g}\left(1\pm\sqrt{1-\frac{3r_g^2m^2c^2}{L^2}}\right).$$

•**[1 Mark]**
The larger root corresponds to the stable orbit.
•**[1 Mark]**
One can see that

$$1-\frac{3r_g^2m^2c^2}{L^2}>0.$$

•**[2 Marks]**
Hence

$$-\sqrt{3}mcr_g\leq L\leq\sqrt{3}mcr_g.$$

Substituting $L=\sqrt{3}mcr_g$ into equation for the radius of circular orbits, we have for the radius of the innermost stable orbit $r_{lso}=3r_g$.

2. (a) **[5 Marks]** Using the equation $ds=0$ with $\theta,\ \phi=\text{const}$, consider the propagation of radial light signals in the Schwarzschild space-time. Consider a photon emitted outward from $r=r_0$ at time $t=0$. Show that the world-line of the photon is given by

$$ct=r-r_0+r_g\ln\frac{r-r_g}{r_0-r_g}.$$

(b) **[10 Marks]** A particle moves along a radial geodesic in the Schwarzschild metric. Using the expression for ds and an appropriate component of geodesic equation, show that if the particle starts to fall freely from infinity, then

$$r(\tau)=\left[r^{3/2}(\tau_0)-\frac{3}{2}cr_g^{1/2}(\tau-\tau_0)\right]^{2/3},$$

where τ is the proper time ($ds=cd\tau$).

(c) **[7 Marks]** A free-falling observer moves radially with zero velocity at infinity in the gravitational field of Schwarzschild black hole. When it passes the radius $r_0\gg r_g$ he starts to send outward radio-pulses with constant rate. The very small time interval between two subsequent pulses measured by clocks of the observer is equal to $\Delta\tau \ll r_g/c < r/c$. The second observer resting very far from the black hole receives signals sent by the first observer. Show that the time interval between the $(n+1)^{th}$ and the n^{th} pulse depends on n according to

$$\Delta t_n = \frac{\Delta\tau}{1-\sqrt{\frac{r_g}{r_n}}},$$

where r_n is the radius at which the n^{th} pulse is emitted.

B2(a)(*seen similar*)

•**[1 Mark]**

From $ds = 0$ for $\theta,\ \phi = const$, we have

$$c^2(1-\frac{r_g}{r})dt^2 - (1-\frac{r_g}{r})^{-1}dr^2 = 0,$$

•**[2 Marks]**

hence

$$cdt = (1-\frac{r_g}{r})^{-1}dr = r(r-r_g)^{-1}dr = \int r(r-r_g)^{-1}dr =$$

$$= \int (r - r_g + r_g)(r-r_g)^{-1}dr = (r-r_g) + r_g \ln(r-r_g) + C.$$

•**[2 Marks]**

If at $t = 0$ $r = r_0$, then

$$C = -[(r_0 - r_g) + r_g \ln(r_0 - r_g)],$$

and finally

$$ct = r - r_0 + r_g \ln\frac{r-r_g}{r_0-r_g}.$$

B2(b)(*seen similar*)

•**[2 Marks]**

A particle moves along radial geodesic in the Schwarzschild metric, then

$$\frac{cd^2t}{ds^2} + \Gamma^0_{00}c^2(\frac{dt}{ds})^2 + 2\Gamma^0_{01}c\frac{dt}{ds}\frac{dr}{ds} + \Gamma^0_{11}(\frac{dr}{ds})^2 = 0.$$

•**[1 Mark]**

$$\Gamma^0_{00} = \frac{1}{2}g^{00}(g_{00,0} + g_{00,0} - g_{00,0}) = 0,$$

•**[1 Mark]**

$$\Gamma^0_{01} = \frac{1}{2}g^{00}(g_{00,1} + g_{10,0} - g_{01,0}) = \frac{1}{2}g^{00}\frac{dg_{00}}{dr} = \frac{1}{2}(1-\frac{r_g}{r})^{-1}\frac{d(1-\frac{r_g}{r})}{dr} = \frac{r_g}{2r^2}(1-\frac{r_g}{r})^{-1},$$

•**[2 Marks]**

$$\Gamma^0_{11} = \frac{1}{2} g^{00}(g_{10,1} + g_{10,1} - g_{11,0}) = 0,$$

so we have

$$\frac{d^2t}{ds^2} + + \frac{r_g}{r^2}(1 - \frac{r_g}{r})^{-1}\frac{dt}{ds}\frac{dr}{ds} = 0,$$

or

$$\frac{dt}{ds}(\frac{dt}{ds}) + (1 - \frac{r_g}{r})^{-1}\frac{dt}{ds}\frac{d}{ds}(1 - \frac{r_g}{r}) = (1 - \frac{r_g}{r})^{-1}\frac{dt}{ds}[\frac{dt}{ds}(1 - \frac{r_g}{r})] = 0,$$

hence

$$\frac{dt}{ds}(1 - \frac{r_g}{r}) = C.$$

•**[1 Mark]**
At infinity $\frac{dt}{ds} = c^{-1}$, hence $C = c^{-1}$.
•**[2 Marks]**
Substituting this into eq. for ds, we have

$$1 = (1 - \frac{r_g}{r})c^2(1 - \frac{r_g}{r})^{-2}c^{-2} - (1 - \frac{r_g}{r})^{-1}(\frac{dr}{ds})^2,$$

$$1 - \frac{r_g}{r} = 1 - (\frac{dr}{ds})^2 \Rightarrow (\frac{dr}{d\tau}) = -c\sqrt{\frac{r_g}{r}},$$

we take "$-$" for falling objects, then

$$\frac{2}{3}r^{3/2}(\tau) - r^{3/2}(\tau_0) = -cr_g^{1/2}(\tau - \tau_0),$$

•**[1 Mark]**
and finally

$$r(\tau) = [r^{3/2}(\tau_0) - \frac{3}{2}cr_g^{1/2}(\tau - \tau_0)]^{2/3}.$$

B2(c)(*unseen*)

•**[1 Mark]**

$$\Delta t_n = \Delta t_1 + \Delta t_2,$$

where Δt_1 is the time spent by the first observer to travel between r_n and r_{n+1}, and Δt_2 is the time spent by the n^{th} pulse to travel between r_{n+1} and r_n.
•**[2 Marks]**

$$\Delta t_1 = \Delta\tau(1 - \frac{r_g}{r})^{-1},$$

•**[2 Marks]**

$$\Delta t_2 = \frac{1}{c}\Delta r(1-\frac{r_g}{r})^{-1} = \sqrt{\frac{r_g}{r}}\Delta\tau(1-\frac{r_g}{r})^{-1},$$

•[**2 Marks**]

hence

$$\Delta t_n = \Delta\tau(1-\frac{r_g}{r})^{-1} + \sqrt{\frac{r_g}{r}}\Delta\tau(1-\frac{r_g}{r})^{-1} = (1+\sqrt{\frac{r_g}{r}})\Delta\tau(1-\frac{r_g}{r})^{-1},$$

hence

$$\Delta t_n = \frac{\Delta\tau}{1-\sqrt{\frac{r_g}{r_n}}}.$$

3. (a) [**10 Marks**] A weak gravitational wave is a small perturbation of the Minkovski metric, $g_{ik} = \eta_{ik} + h_{ik}$. Show that $g^{ik} = \eta^{ik} - \eta^{in}\eta^{km}h_{nk}$. Use a linear coordinate transformation

$$x'^i = x^i + \xi^i,$$

where ξ^i are small functions of x^i, to impose on h_{ik} the following four supplementary conditions

$$\eta^{km}h_{mi,k} - \frac{1}{2}\delta^k_i\eta^{nm}h_{nm,k} = 0.$$

Show that after such transformation the Ricci tensor is reduced to

$$R_{ik} = -\frac{1}{2}\eta^{lm}\frac{\partial^2 h_{ik}}{\partial x^l \partial x^m}.$$

(b) [**5 Marks**] Consider a ring of test particles initially at rest in the (y,z)-plane, perturbed by a plane monochromatic gravitational wave propagating in x-direction with frequency ω and amplitude h_0. Explain what is meant by "+" and "×" polarizations. Sketch the shape of the ring at $x=0$ and at times $t = 0, \frac{\pi}{2\omega}, \frac{\pi}{\omega}, \frac{3\pi}{2\omega}$ and $\frac{2\pi}{\omega}$ for two different polarizations of the gravitational wave: (i) $h_+ = h_0 \sin\omega(t-x/c)$, $h_\times = 0$; and (ii) $h_+ = 0$, $h_\times = h_0\sin\omega(t-x/c)$.

(c) [**7 Marks**] Two bodies of equal mass $m_1 = m_2 = m$, attracting each other according to Newton's law, move in circular orbits around their common centre of mass with orbital period P. Using the quadrupole formula for the generation of gravitational waves show that in order of magnitude

$$h \sim \frac{r_g}{R}\left(\frac{r_g}{cP}\right)^{2/3},$$

where R is the distance to the system and $r_g = \frac{2Gm}{c^2}$ is the gravitational radius.

B3(a)(*book work*)

•[**3 Marks**]

 [This question continues overleaf . . .]

If $g_{ik} = \eta_{ik} + h_{ik}$, where h_{ik} are small, contravariant metric tensor can be written as $g^{ik} = \eta^{ik} + a^{ik}$, where a^{ik} are also small. Taking into account that $g_{ik}g^{kn} = \delta_i^n$ we have

$$(\eta_{ik} + h_{ik})(\eta^{kn} + a^{kn}) = \delta_i^n,$$
$$\delta_i^n + \eta_{ik}a^{kn} + h_{ik})\eta^{kn} = \delta_i^n,$$
$$\eta_{ik}a^{kn} = -h_{ik}\eta^{kn},$$
$$\eta^{im}\eta_{ik}a^{kn} = -\eta^{im}h_{ik}\eta^{kn},$$
$$\delta_k^m a^{kn} = -\eta^{im}\eta^{kn}h_{ik},$$
$$a^{mn} = -\eta^{mi}\eta^{nk}h_{ik},$$

or

$$a^{ik} = -\eta^{in}\eta^{km}h_{nk}.$$

•**[2 Marks]**

Writing the Rieman and Ricci tensors in linear approximation we have

$$R_{iklm} = \frac{1}{2}\left(\frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{km}}{\partial x^i \partial x^l}\right),$$

and

$$R_{ik} = \frac{1}{2}(-\eta^{lm}\frac{\partial^2 h_{ik}}{\partial x^l \partial x^m} + \eta^{lm}\frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \eta^{lm}\frac{\partial^2 h_{km}}{\partial x^i \partial x^l} - \eta^{lm}\frac{\partial^2 h_{lm}}{\partial x^i \partial x^k}).$$

•**[1 Mark]**

We have four arbitrary functions ξ, thus we can impose on h_{ik} four supplementary conditions:

$$\eta^{km}h_{mi,k} - \frac{1}{2}\delta_i^k\eta^{nm}h_{nm,k} = 0,$$

•**[4 Marks]**

then

$$R_{ik} = \frac{1}{2}\left(-\eta^{lm}\frac{\partial^2 h_{ik}}{\partial x^l \partial x^m} + \eta^{lm}h_{km,l,i} + \eta^{lm}h_{im,l,k} - \eta^{lm}h_{lm,k,i}\right) =$$
$$= \frac{1}{2}\left(-\eta^{lm}\frac{\partial^2 h_{ik}}{\partial x^l \partial x^m} + \frac{1}{2}\delta_i^l\eta^{lm}h_{lm,l,k} + \frac{1}{2}\delta_k^l\eta^{lm}h_{lm,l,i} - \eta^{lm}h_{lm,k,i}\right) =$$
$$= \frac{1}{2}\left(-\eta^{lm}\frac{\partial^2 h_{ik}}{\partial x^l \partial x^m} + \eta^{lm}h_{km,l,i} + \eta^{lm}h_{im,l,k} - \eta^{lm}h_{lm,k,i}\right) =$$
$$= \frac{1}{2}\left(-\eta^{lm}\frac{\partial^2 h_{ik}}{\partial x^l \partial x^m} + \frac{1}{2}\delta_i^l\eta^{lm}h_{lm,l,k} + \frac{1}{2}\delta_k^l\eta^{lm}h_{lm,l,i} - \eta^{lm}h_{lm,k,i}\right) =$$
$$= \frac{1}{2}\left(-\eta^{lm}\frac{\partial^2 h_{ik}}{\partial x^l \partial x^m} + \frac{1}{2}\eta^{lm}h_{lm,i,k} + \frac{1}{2}\eta^{lm}h_{lm,k,i} - \eta^{lm}h_{lm,k,i}\right) = -\frac{1}{2}\eta^{lm}\frac{\partial^2 h_{ik}}{\partial x^l \partial x^m}.$$

B3(b)(*seen similar*)

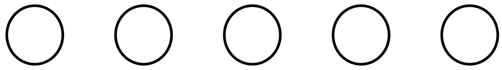
 [This question continues overleaf ...]

•**[1 Mark]**

By transformation of coordinates it is possible to eliminate all components of h_{ik} except transverse components $h_{22} = -h_{33} = h_+$ and $h_{23} = h_\times$. The two independent components h_+ and $h_\times$ are called $+$ and $\times$ polarizations.

•**[2 Marks]**

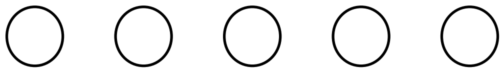
i)



$t = 0$ $t = T/4$ $t = T/2$ $t = 3T/4$ $t = T$

•**[2 Marks]**

ii)



$t = 0$ $t = T/4$ $t = T/2$ $t = 3T/4$ $t = T$

B3(c)(*unseen*)

•**[2 Marks]**

To an order of magnitude and omitting indices we have

$$h \sim \frac{G}{c^4 R}\ddot{D} \sim \frac{G}{c^4 R} m r^2 P^{-2}.$$

•**[2 Marks]**

Taking into account that according to Newton low

$$P^{-2} \sim G m r^{-3}$$

we have

$$r \sim (G m P^2)^{1/3},$$

•**[3 Marks]**

hence

$$h \sim \frac{Gm}{c^4 R P^2}\left(G m P^2\right)^{2/3} \sim \frac{r_g}{c^2 R P^2}\left(r_g c^2 P^2\right)^{2/3} \sim \frac{r_g}{R}\left(\frac{r_g}{cP}\right)^{2/3}.$$

[End of examination paper.]