11 Inflationary Cosmology

A Note on Initial Conditions: It is important to appreciate that the flatness and horizon 'problems' are problems with the initial conditions in the early universe. Within the context of the big bang framework as we have discussed it thus far, there is nothing preventing us from setting the initial conditions in such a way that the universe is indeed as close to spatial flatness at nucleosynthesis (or some earlier time) as is required to be consistent with our observations. Likewise, we can choose the initial conditions so that the temperature of the microwave background is uniform. These initial conditions are possible – the problem is, are they natural or plausible? Such a sensitive restriction as shown in Eq. (9.18) suggests there is more physics to be considered.

11.1 Definition of Inflation

The flatness problem arose because $\dot{a}$ decreases with time in the big bang model due to the attractive nature of gravity, thereby causing the Ω–parameter to evolve away from unity. The problem is a problem of understanding why Ω is still close to unity at the present epoch. However, if the expansion were to accelerate for some period of time, $\dot{a}$ would increase and this would cause the value of Ω to move towards unity. The longer such an epoch of accelerated expansion were to last, the closer Ω would tend towards unity.

In general, the definition of *inflation* is just such an epoch of *accelerated* expansion:

$$\frac{d^2a}{dt^2} > 0 \tag{11.1}$$

Suppose for the moment that the universe underwent an inflationary expansion at some very early time, starting at a time t_b and ending at a time t_f, where $t_f \ll 1\,\mathrm{sec}$. The effect of introducing such a finite era of accelerated expansion on the evolution of Ω is shown in Fig. (11.1). Before inflation starts, $t < t_b$, Ω moves away from unity. This behaviour is reversed between $t_b < t < t_f$ and Ω tends towards unity. After the accelerated expansion has ended, Ω again moves away from unity for $t > t_f$ until the present epoch, t_0. The value of Ω today is determined by how close Ω was to unity at the end of inflation and this in turn is determined by how long inflation lasted in the early universe. A longer period of inflation implies that Ω_0 is closer to unity today. So we see that if the accelerated expansion lasted long enough, Ω would have been pushed so close to unity during inflation that it would still be extremely close to this value today, some ten billion years later. Thus, in principle, *the flatness problem can be solved by an epoch of accelerated, inflationary expansion.*

Given this qualitative observation, we must now address three, key questions:

1. What causes the universe to undergo a period of accelerated expansion at some very early time?

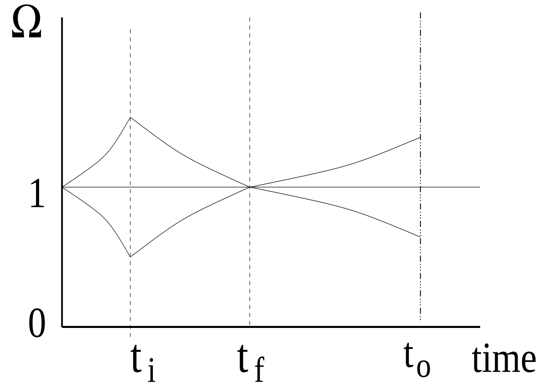

Fig. (11.1). Illustrating how inflation can solve the flatness problem of the hot big bang model. Suppose the universe undergoes an epoch of accelerated expansion – inflation – starting at a time t_b and ending at a time t_f. Before inflation begins, the Ω–parameter moves away from unity, as explained in Section 9.2.1. However, due to the accelerated nature of the expansion during inflation, Ω moves towards unity between t_b and t_f. After inflation has come to an end, Ω once more moves away from unity because the expansion is now decelerating. The value of Ω today is determined by how close Ω was to unity at the end of inflation and this in turn is determined by how long inflation lasts in the early universe. A longer period of inflation implies that Ω_0 is closer to unity today. Hence, the flatness problem – that is, the problem of understanding why Ω_0 is still close to unity at the present epoch – can be resolved if the universe underwent a sufficiently long epoch of accelerated expansion in its most distant past. It is important to emphasize that if $\Omega < 1$ initially, Ω is always less than unity and similarly if $\Omega > 1$ initially, Ω is always greater than unity. (If Ω were to pass through unity this would be equivalent to changing the sign of the constant k in the Friedmann equation (2.27), but this constant is fixed for all time). Furthermore, recall that if $\Omega = 1$ precisely, then $\Omega = 1$ for all time, since this corresponds to $k = 0$.

2. How long does inflation have to last to solve the flatness problem?
3. How can inflation be made to end? We know from our understanding of primordial nucleosynthesis that the end of inflation must have occurred well before the universe was one second old.

The answer to the first question follows from the acceleration equation (2.29). We see immediately that a necessary (and sufficient) condition for inflation is that the energy density and pressure must satisfy

$$p < -\frac{\rho c^2}{3} \tag{11.2}$$

Since ordinary matter has positive density, this implies that the pressure must be *negative*. We have already discussed in the Section 6 how condition (11.2) can be satisfied if there is a cosmological constant present in the universe. There, we alluded to the idea that such a term can be viewed as a special type of matter with an equation of state where $\gamma = 0$, i.e.,

$$p = -\rho c^2 \tag{11.3}$$

Let us suppose that there is a particle (we discuss its origin in the next subsection) that has an equation of state given by Eq. (11.3). It follows from the conservation equation that the density of this matter stays constant with time, $\rho_{\text{particle}} \equiv V =$ constant. It soon dominates any other type of matter in the universe, since the density of relativistic and non–relativistic matter decreases as $\rho_{\text{rel}} \propto a^{-4}$ and $\rho_{\text{nonrel}} \propto a^{-3}$, respectively. The curvature term also becomes negligible. Thus, the Friedmann equation becomes

$$H^2 = \frac{8\pi G}{3}\left(\rho_{\text{rel}} + \rho_{\text{nonrel}} + V\right) - \frac{kc^2}{a^2} \rightarrow \frac{8\pi G V}{3} = \text{constant} \tag{11.4}$$

Thus,

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G V}{3} \qquad \Longrightarrow \qquad a \propto \exp\left(\sqrt{\frac{8\pi G V}{3}}t\right) \tag{11.5}$$

and the expansion is *exponential*, as we saw previously in Section 6.3. Moreover, it is necessarily accelerating, $\ddot{a} = (8\pi G V/3)a > 0$. Hence, we have answered the first question.

In the simplest models of inflation, the universe becomes dominated by the (constant) density of such a particle for a very brief time. Typically inflation starts at the extraordinarily early time of $t_b \approx 10^{-36}$ s and ends at a time $t_f \approx 10^{-34}$ s. During this time the scale factor varies as

$$a(t) = a_f e^{\beta(t-t_f)}, \qquad \beta \equiv \sqrt{\frac{8\pi G V}{3}} \tag{11.6}$$

where $a_f \equiv a(t_f)$ represents the value of the scale factor at the end of inflation. Inflation ends when the particle decays – it is at this point that the energy contained within the particle is then released and converted into ordinary relativistic particles. In some sense, this defines the starting point of the radiation–dominated, big bang model.

One word of warning: the 'effective' cosmological constant, V, that is responsible for inflation should *not* be confused with the cosmological constant that appears to be dominating the universe at the present time, as inferred from supernovae observations (Section 6.4). Although the physics may be similar, the energy scales are very different. The cosmological constant that causes inflation in the early universe has a typical density of the order $V \approx 10^{80}\,\text{kg}\,\text{m}^{-3}$. The cosmological term present today has a density comparable to the critical density (4.5), $\rho_c \approx 10^{-26}\,\text{kg}\,\text{m}^{-3}$ and this is very much smaller.

11.2 Solving the Flatness Problem with Inflation

From our discussion in Section 9.2.1, we require enough inflation so that $|\Omega_{\text{nuc}} - 1| = 10^{-17}$ at the epoch of nucleosynthesis, $t_{\text{nuc}} = 1\,\text{sec}$ [Eq. (9.17)]. The key step is to note from the Friedmann equation (4.4) that the quantity $|\Omega - 1|a^2H^2$ is a constant. Hence, it is independent of time and we may write

$$|\Omega_b - 1|\, a_b^2 H_b^2 = |\Omega_f - 1|\, a_f^2 H_f^2 \tag{11.7}$$

where subscripts b and f denote that quantities are to be evaluated at the beginning and end of inflation, respectively. We assume that the value of Ω at the start of inflation is not too far from unity, i.e., $|\Omega_b - 1| = \mathcal{O}(1)$. This is justified since inflation occurs in the very early universe and we know that curvature is negligible at sufficiently early times. Moreover, since the expansion is exponential during inflation, the Hubble parameter is constant and we may write $H_b = H_f$. Thus, Eq. (11.7) simplifies to

$$\frac{a_f}{a_b} \approx \frac{1}{\sqrt{|\Omega_f - 1|}} \tag{11.8}$$

This expression relates how much inflation occurs to the value of the Ω–parameter at the end of inflation.

The idea is that the universe becomes radiation dominated after inflation, so the scale factor grows as $a \propto t^{1/2}$ for $t > t_f$. However, from the Friedmann equation (4.4) we may relate the value of Ω at the end of inflation to its value at nucleosynthesis:

$$\frac{|\Omega_f - 1|}{|\Omega_{\rm nuc} - 1|} = \frac{a^2_{\rm nuc} H^2_{\rm nuc}}{a^2_f H^2_f} = \frac{\dot{a}^2_{\rm nuc}}{\dot{a}^2_f} = \frac{t_f}{t_{\rm nuc}} \tag{11.9}$$

Thus, by substituting the required value of $\Omega_{\rm nuc}$ from Eq. (9.17) into Eq. (11.9), we find that the amount of inflation that must occur in order for Ω to be sufficiently close to unity at nucleosynthesis must exceed

$$\frac{a_f}{a_b} \approx 10^9 \left(\frac{t_f}{\rm sec}\right)^{-1/2} \tag{11.10}$$

This is determined by the time at which inflation ends. If inflation ends at $t_f \approx 10^{-34}$ s, then this implies that the expansion factor of the universe during inflation must be *huge*:

$$\frac{a_f}{a_b} > 10^{26} \tag{11.11}$$

Compare this with the factor by which the universe has expanded since the time of primordial nucleosynthesis through to the present epoch. Nevertheless, inflation can claim to have solved the flatness problem if the universe did indeed inflate by the amount given in Eq. (11.11).

11.3 Solving the Horizon Problem with Inflation

How does inflation solve the horizon problem? Recall from Section 9.2.2 that the horizon problem arises when we attempt to understand why the cosmic microwave background has a (very nearly) uniform temperature today. This is a problem because radiation reaching us today from opposite regions of the universe was separated by more than the horizon distance at the epoch of decoupling when the matter and radiation began to evolve separately. Formally, we expressed the problem by saying that the rescaled horizon distance (9.24) evaluated at the decoupling era was less than twice the present horizon distance, i.e.,

$$a_0 \int_0^{t_{\rm dec}} \frac{cdt}{a(t)} < 6ct_0 \quad \Longrightarrow \quad \text{Horizon} \quad \text{Problem} \tag{11.12}$$

By implication, the problem would not arise if this inequality were to be reversed:

$$a_0 \int_0^{t_{\rm dec}} \frac{cdt}{a(t)} > 6ct_0 \tag{11.13}$$

We will now show that Eq. (11.13) is indeed satisfied if sufficient inflation occurred to solve the flatness problem.

Firstly, we note that if the stronger condition

$$a_0 \int_{t_b}^{t_f} \frac{c}{a(t)} dt > 6ct_0 \tag{11.14}$$

is satisfied, then Eq. (11.13) is also satisfied, since the range of integration is smaller in Eq. (11.14). Hence, it is sufficient to show that Eq. (11.14) is satisfied to resolve the horizon problem.

Evaluating the integral (11.14) implies that

$$\frac{a_0}{a_f}\left(e^{\beta(t_f - t_b)} - 1\right) > 6\beta t_0 \tag{11.15}$$

where we assume that Eq. (11.6) applies between the limits of integration, t_b and t_f. Let us further assume that for most of its history since the end of inflation, the universe has been dominated by non–relativistic (pressureless) matter:

$$\frac{a_0}{a_f} = \left(\frac{t_0}{t_f}\right)^{2/3} \tag{11.16}$$

Substituting Eqs. (11.6) and (11.16) into Eq. (11.15) implies that

$$\frac{a_f}{a_b} > 1 + 6\beta t_0 \left(\frac{t_f}{t_0}\right)^{2/3} = 6\beta t_f \left(\frac{t_0}{t_f}\right)^{1/3} \tag{11.17}$$

and for the order–of–magnitude estimates $t_f \approx 10^{-34}\,$s and $t_0 \approx 10^{17}\,$s for the end of inflation and present age of the universe, respectively, this becomes

$$\frac{a_f}{a_b} > 1 + 6 \times 10^{17} \beta t_f > 1 + 6 \times 10^{17} \beta (t_f - t_b) = 1 + 6 \times 10^{17} \ln\left(\frac{a_f}{a_b}\right) \tag{11.18}$$

However, the condition to solve the flatness problem is given by Eq. (11.11), and if this is satisfied, then Eq. (11.18) is *automatically* satisfied, as can be verified by direct substitution of Eq. (11.11). In other words, *inflation solves both the horizon problem and flatness problems simultaneously.* This answers the second question.

11.4 Solving the Relic Particle Problem with Inflation

How is the problem of relic particles solved? If inflation starts after these particles are formed, the huge exponential increase in volume implies that their density becomes negligible exponentially quickly. Indeed, if the flatness problem is solved by satisfying Eq. (11.11), the density decreases by the factor $(a_f/a_b)^3 > 10^{78}$. The monopole problem is therefore trivially solved if the flatness and horizon problems are solved.

11.5 Predicted Value of Ω_0 from Inflation

Inflation leads to a predicted value for the Ω–parameter at the present epoch, Ω_0. Returning once more to the Friedmann equation (4.4), we may take the time–independent quantity $|\Omega - 1|a^2H^2$ and evaluate it at the onset of inflation and the present–day:

$$|\Omega_b - 1|a_b^2 H_b^2 = |\Omega_0 - 1|a_0^2 H_0^2 \tag{11.19}$$

where quantities on the left hand side are evaluated at the start of inflation and those on the right hand side at the present epoch. Thus,

$$|\Omega_0 - 1| = |\Omega_b - 1| \left(\frac{a_b}{a_f}\right)^2 \left(\frac{a_f}{a_0}\right)^2 \left(\frac{H_f}{H_0}\right)^2 \tag{11.20}$$

where we note that $H_b = H_f$ since the expansion during inflation is exponential. Let us once more assume the universe is dominated by pressureless matter since the end of inflation, i.e., that Eq. (11.16) holds. Moreover, after inflation has come to an end, $H \propto t^{-1}$, so $H_0/H_f = t_f/t_0 \approx 10^{-34}/10^{17} \approx 10^{-51}$. Bringing all this together implies that the present day value of Ω_0 is related to the amount of inflation by

$$|\Omega_0 - 1| \approx 10^{34}|\Omega_b - 1| \left(\frac{a_f}{a_b}\right)^{-2} \tag{11.21}$$

For the horizon and flatness problems to be solved requires $a_f/a_b > 10^{26}$ so we see that

$$|\Omega_0 - 1| < 10^{-18}|\Omega_b - 1| \tag{11.22}$$

Thus, unless $|\Omega_b - 1|$ is unrealistically large, the huge exponential expansion of inflation implies that Ω_0 should be extremely close to unity at the present time. This is a cornerstone prediction of inflation. Given the uncertainties in measuring Ω_0, it implies that *we should effectively observe* $\Omega_0 = 1$ *today.* Failure to make such an observation would rule out inflation as a viable model of the very early universe.

In summary, it is important to emphasize that one modification to the big bang model, namely an era of accelerated (exponential) expansion at a very early time, allows us to simultaneously solve the *three* problems of the big bang scenario that we discussed in Section 9.2. A key prediction is that we should measure Ω_0 to be very close to unity today.

11.6 Scalar Fields and Quantum Fluctuations

A complete understanding of the microphysical origin of inflationary cosmology requires some technical knowledge of particle physics theory and unified theories of the fundamental interactions, such as Grand Unified Theories and Superstring Theories. These topics are beyond the scope of this course. However, we can gain some insight into the physics of inflation by means of a qualitative discussion and some useful analogies. This is the purpose of the present subsection. This is important because it illustrates how particle physics has important implications for cosmology, and vice–versa.

The answer to the third question posed in Section 11.1, namely what is the mechanism that causes inflation to end, is found in certain particles known as *scalar fields*. Scalar particles are predicted to exist in many particle physics theories. One of their key properties is that they have *zero* spin (angular momentum). (Recall the electron has a spin $\hbar/2$ and the photon a spin of $1\hbar$). A scalar field is denoted by the symbol ϕ. Mathematically, it should be viewed as a function of space and time that contains information about the density and pressure of the particle at every point in the universe. More heuristically, the value of ϕ provides a measure of the number of scalar particles present in the universe.

Scalar particles move and so have a kinetic energy. This is given by $\frac{1}{2}\dot{\phi}^2$. (Compare this with that of a particle of unit mass). In general, scalar particles can only interact through gravity and with themselves. Indeed, typically, a scalar particle can decay into one or more identical scalar particles. This production of new particles is made possible because a scalar field also has a potential energy associated with it. This is released when the particle decays and is converted into the kinetic energy and mass–energy of the new particles. The potential energy is usually denoted as a function of the field, $V(\phi)$. Hence, the total density associated with a scalar field is then

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{11.23}$$

Now, since the scalar particle has both a kinetic and potential energy, its dynamics is similar to that of a ball rolling down a hill. Indeed, we may think of the scalar particle as such a ball, where the value of the field ϕ denotes the position of the ball on its potential and the shape of the hill corresponds to the form of the function $V(\phi)$. This is shown schematically in Fig. (11.2).

The specific shape of the hill/potential is determined by the specific nature of the theory that describes the particle physics. Nevertheless, in general it exhibits a number of important, characteristic features. There is a valley at the bottom, as denoted by the point O. A plateau region lies to the right of O and ends when it meets a steep cliff. This plateau is inclined slightly to the horizontal.

If we were to hold a ball at rest somewhere on the plateau, it would have a certain amount of potential energy associated with it. This would be determined by its height above the point O. If we were to then release this ball, it would begin to roll slowly

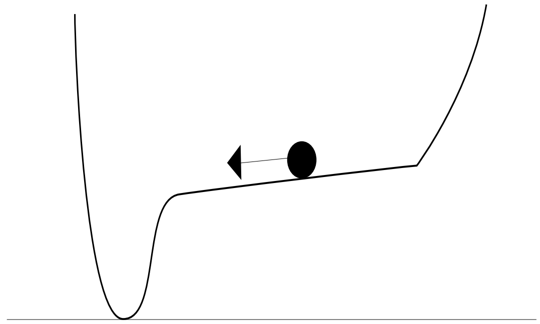

Fig. (11.2). The energy density of the universe during inflation may be viewed in terms of the potential energy associated with a ball that rolls slowly down a hill. Inflation occurs when the ball is located on the plateau region and ends when it falls down the valley towards O.

down the hill towards the valley. Its potential energy would be converted into kinetic energy in the process. Eventually, the ball would reach the edge of the plateau and then fall rapidly into the bottom of the valley. Its remaining potential energy would be quickly converted into kinetic energy.

The idea is that the potential energy of the ball represents that of the scalar field in the early universe. This potential remains almost constant when it is located on the plateau region, although it does fall slightly as the ball/scalar field slowly rolls to the left. Thus, the kinetic energy of the scalar field is negligible and, since it is effectively constant, the potential of the field soon comes to dominate the density of the universe. As we have seen, a constant density acts as a cosmological constant and results in inflation.

The end of inflation occurs when the ball reaches the end of the plateau and rushes down towards O. The conversion of the ball's potential energy into kinetic energy represents the conversion of the potential energy of the scalar field into ordinary particles and radiation. The scalar field effectively decays at this point. A vast amount of energy is released in this decay process and the newly created particles were very energetic. Hence the temperature after inflation was very high and may have been as high as 10^{28} K. Thus, the conditions in the universe immediately after inflation would have resembled those of the hot big bang. The history of the universe from this time onwards can then be described within the context of the big bang model.

We have now addressed the third question raised in Section 11.1, namely the origin of the cosmological constant responsible for inflation and the process whereby inflation is brought to an end. *The cosmological constant arises through the potential energy of a scalar field and inflation ends when this field decays into matter and radiation.* Such a decay corresponds to the release of the scalar field's potential energy into mass–energy and momentum of ordinary matter and radiation.

The picture of inflation that we have outlined above is incomplete because it does not take into account quantum fluctuations. Intrinsic uncertainties arise in *all* physical processes, but these uncertainties only become important on very small scales. Since inflation occurred at a very early time, the universe would have been tiny in comparison to its present size. It is reasonable to suppose that quantum fluctuations would have played a significant role.

One consequence of these quantum fluctuations is that the energy and position of a particle can never be precisely measured. This follows as a direct consequence of Heisenberg's uncertainty principle:

$$\Delta p \Delta x \geq \hbar \tag{11.24}$$

where Δp and Δx represent the uncertainties in momentum and position, respectively. This same principle will apply to the ball/scalar field as it travels along the plateau of Fig. (11.2). We may understand how the inflationary process is modified by investigating how the motion of this ball is affected.

Suppose the size of this ball is comparable to that of a typical elementary particle. In this case, the uncertainties in the energy and speed of the ball are significant. These uncertainties will influence the motion of the ball by a tiny amount as it moves along the plateau. They operate in random directions and sometimes the ball will be pushed slightly towards the cliff. On other occasions, it is directed downwards by the fluctuations. The position of the ball on the plateau will be uncertain to some extent. The overall result is that the ball may move towards the valley at a faster or slower rate than we previously thought.

What does this mean for the inflating universe? As we have seen, inflation proceeds when the ball is on the plateau and it ends when the ball falls over the edge and rushes down into the valley. If the ball is slightly higher up the plateau, it will reach the valley at a *later* time than expected. Inflation will last a little bit longer. Conversely, inflation will end sooner if the quantum fluctuations push the ball downwards. In short, the overall effect of the quantum fluctuations is that they cause inflation to end *at different times in different regions of the universe.*

This time difference has important consequences. It implies that different regions of the universe inflate by different amounts. Hence, the density of matter will vary throughout the universe after inflation. Some regions will be denser than others. The quantum fluctuations that act on the ball are very weak, so the density variations will be small. Nevertheless, they do play a significant role during the subsequent evolution of the universe.

The irregularities in the density of the universe after inflation would have affected its temperature. The denser regions of the universe would have had a slightly higher temperature than the less dense regions. These differences in temperature between the high and low density regions survived as the universe expanded. They were still present when the matter and radiation stopped interacting directly with one another at the decoupling era. The cosmic radiation from the high density regions would have had a slightly higher temperature than the average at that time. Conversely, radiation from the low density areas would have been slightly cooler.

As we have emphasized previously, this radiation has remained essentially undisturbed since the epoch of decoupling. Only its wavelength has altered, because the universe has continued to expand. The temperature differences should still be present today in the cosmic microwave background. In other words, the radiation that comes from one part of the universe should have a slightly different temperature than radiation that arrives from another part. This is what is observed – the cosmic microwave background is very uniform but small irregularities are indeed measured to one part in 10^5. In other words, the temperature difference, ΔT, relative to the average, T_0, is

$$\frac{\Delta T}{T_0} \approx 10^{-5} \tag{11.25}$$

The temperature anisotropies in the cosmic microwave background are discussed in more detail in Section 13.

A second important consequence of the quantum fluctuations and subsequent density irregularities is that they provide the initial perturbations necessary to form galaxies and clusters of galaxies. A higher density in a given volume of space implies that more matter is present than in the surrounding regions. Consequently, even more matter would have been attracted from the surroundings into these high density regions due to the enhanced attractive pull of gravity. The regions of high density became even denser, whilst the surroundings became effectively devoid of matter. Eventually, the gravitational attraction of the matter in the denser regions became more important than the outward effects of the cosmic expansion. At this point, the regions began to behave as separate, gravitationally bound objects. It is these regions that we identify today as galaxies.

This is a remarkable feature of inflation. It implies that the largest structures in existence today may have arisen out of processes that occurred on the smallest of scales when the universe was just a fraction of a second old. This suggests that the idea of inflation could be tested experimentally. In principle, we may employ the theory to predict how the universe should look if galaxies indeed grew out of these quantum fluctuations. This prediction may then be compared to the observations. If the idea of inflation is correct, theory and observation should agree at some level.

The implications of this are profound. Recall that from the temperature–time relation (8.34), we concluded that the energy scales of physical processes before the universe was 10^{-12} s old were beyond the reach of particle accelerators, implying that we could never experimentally explore beyond this barrier. In particular, it implies that the physics of the universe when it was 10^{-34} s old could never be tested. Inflation reverses this somewhat pessimistic conclusion, because it relates the structure of the universe when it was just 10^{-34} seconds old directly to the structure of the universe when it is ten billion years old – the universe may therefore provide the ultimate laboratory for probing physics on very small scales.

In view of the fundamental relationship between quantum fluctuations and the formation of galaxies that we have outlined above, we proceed in the following Section to investigate further how inhomogeneities in the density of matter grew in the universe.