12 Origin of Structure in the Universe

12.1 Growth of Inhomogeneities in a Pressureless Universe

Let us consider a spherically symmetric sphere of matter whose density is slightly higher than the average density in the universe. This sphere wants to collapse. However, it also has a tendency to expand due to the expansion of the universe. Thus, there are two competing effects that determine what happens. If the expansion of the universe is too rapid, the density of the sphere will not grow, that is, the inhomogeneity in the matter distribution will not increase. Conversely, the density can increase if the expansion rate of the universe is sufficiently low. In this case, the sphere will ultimately behave as a self–gravitating, bound object.

We first determine how this sphere of matter evolves when the universe is dominated by pressureless matter, $p = 0$. Now, in Section 2.4 we derived the acceleration equation (2.29) starting from the assumption that the universe could be regarded as a sphere of expanding matter (cf Fig. (2.1)). From that analysis, we can now see that the same equation determines how our sphere of matter behaves in an expanding universe. If the sphere has a radius R, with an initial density ρ and pressure p, it follows that its change in size is given by the acceleration equation (2.29), but with the scale factor substituted for the size of the sphere:

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{12.1}$$

Consider the sphere a short time later when its radius has changed to R' and its density to ρ'. We may relate the new density ρ' to the original density ρ by defining a parameter δ such that

$$\rho' \equiv \rho(1+\delta) \tag{12.2}$$

where

$$\delta \equiv \frac{\rho' - \rho}{\rho} \equiv \frac{\delta\rho}{\rho} \ll 1 \tag{12.3}$$

represents the *density perturbation* and is initially very small. What we are interested in is how δ varies with time as the universe expands.

Now, since the mass of the sphere is conserved, we may write

$$\rho' R'^3 = \rho R^3 \tag{12.4}$$

Substituting in Eq. (12.2) then implies that the perturbed radius is

$$R' = R\left(1 - \frac{\delta}{3}\right) \tag{12.5}$$

where we have expanded the right hand side to first–order in the Taylor series.

Since Eq. (12.1) holds for all times, we may also write that

$$\frac{\ddot{R}'}{R'} = -\frac{4\pi G}{3}\rho' \tag{12.6}$$

(remembering that the pressure vanishes). Hence, substituting for ρ' and R' via Eqs. (12.2) and (12.5) implies that

$$\ddot{R}\left(1 - \frac{\delta}{3}\right) - \frac{2}{3}\dot{R}\dot{\delta} - \frac{R}{3}\ddot{\delta} = -\frac{4\pi G}{3}\rho R\left(1 + \frac{2\delta}{3}\right) \tag{12.7}$$

Eliminating $\ddot{R}$ with Eq. (12.1) then implies that the evolution of the density perturbation, δ, is determined by the second–order differential equation

$$\ddot{\delta} + 2\frac{\dot{R}}{R}\dot{\delta} = 4\pi G\rho\delta \tag{12.8}$$

During the era where the pressure is negligible, $R \propto t^{2/3}$ since the sphere expands in proportion to the scale factor of the universe (recall that the cosmic expansion can be interpreted as a stretching of the space between two points). From the Friedmann equation, the time–dependence of the density is therefore given by

$$H^2 = \frac{\dot{R}^2}{R^2} = \frac{8\pi G\rho}{3} \quad \Longrightarrow \quad \rho = \frac{1}{6\pi G t^2} \tag{12.9}$$

where we have ignored the effects of spatial curvature for simplicity. Substituting Eq. (12.9) into Eq. (12.8) then implies that

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0 \tag{12.10}$$

To solve Eq. (12.10) we 'guess' a power law solution of the form $\delta \propto t^m$ for some constant m. Substituting this into Eq. (12.10) implies that solutions indeed have this form if m takes the values

$$3m^2 + m - 2 = 0 \qquad \Longrightarrow \qquad m = \frac{2}{3} \quad \text{or} \quad m = -1 \tag{12.11}$$

Hence, the most general solution to Eq. (12.10) is

$$\delta = At^{2/3} + Bt^{-1} \tag{12.12}$$

where A and B are arbitrary integration constants. The B term soon decays away as time proceeds and this implies that at later times (large t), the perturbation grows as $t^{2/3}$, i.e., *the perturbation grows at the same rate as the scale factor of the universe.*

$$\delta(t) \propto a(t) \propto t^{2/3} \tag{12.13}$$

This property is very important for what follows. δ measures the density of the overdense region relative to the average density in the universe. As the universe expands the region also expands but becomes progressively denser relative to the background density until its self–gravity dominates any outward pull due to the cosmic expansion. At this point, it stops expanding and begins to recollapse. A rough estimate for when a region splits from the cosmic expansion is when $\delta \approx 1$. Very roughly, we can say that large–scale structure in the universe in the form of galaxies and clusters of galaxies would have formed by the time $\delta \approx 1$, since when this condition is attained, the difference in the density of the region and the background density is larger that the magnitude of the background density itself [see Eq. (12.3)].

12.2 Growth of Inhomogeneities in a Radiation–Dominated Universe

How does δ vary when the universe is dominated by radiation? Let us assume initially that the matter and radiation interact only through gravity, i.e., that there is no pressure due to scattering of radiation off matter. In other words, the matter and radiation are *uncoupled*. The analysis is similar to that of a pressureless universe, but there are two crucial differences. Firstly, the region now expands as $R \propto t^{1/2}$, i.e., $\dot{R}/R = 1/2t$, since this is the behaviour appropriate for a radiation dominated universe. Secondly, the average density of matter can be assumed to be negligible relative to the radiation, so we can take the right–hand side of Eq. (12.8) to vanish. During radiation domination the perturbation therefore evolves according to the simplified equation

$$\ddot{\delta} + \frac{1}{t}\dot{\delta} = 0 \tag{12.14}$$

and the solution to this equation is

$$\delta = A + B \ln t \tag{12.15}$$

Consequently, the growth of the density perturbation in a universe dominated by radiation is at most logarithmic and this is a very slow rate of growth indeed. In effect, *the perturbation is held roughly constant during a radiation dominated phase.* This can be understood from a physical point of view, because the expansion of the universe tends to slow down the rate at which the inhomogeneity can grow. The faster the rate of cosmic expansion the slower the increase of the density perturbation. For a radiation dominated universe the two effects effectively cancel each other out.

The upshot of all this is that in the case where the universe contains a mixture of pressureless matter and radiation that evolve independently from one another (i.e. they are uncoupled), the density perturbation is held close to its initial value as long as the radiation dominates and then grows in direct proportion to the scale factor of the universe once the pressureless matter dominates.

12.3 Galaxy Formation and Dark Matter

However, there is a complication to this picture that we need to account for. In a universe containing baryonic matter and electrons, the temperature of the universe before the epoch of decoupling is sufficiently high to prevent atoms from forming. Consequently, the matter in the universe consists of electrically charged particles and radiation. As we have seen in Section 8.3, the charged matter and radiation will interact electromagnetically. The matter still wants to collapse around regions of high density due to its gravitational attraction. On the other hand, the radiation resists this tendency to collapse and provides an outward pressure that acts on the matter. This pressure is sufficient to prevent any growth in the density perturbation. Hence, the density perturbations in baryons can not begin to grow until after naked electric charge has disappeared from the universe and this does not happen until the epoch of decoupling when the temperature has fallen sufficiently for electrons and protons to combine and form neutral hydrogen.

Now, since a region does not start to behave as a separate entity until $\delta \approx 1$, we require δ to have grown to this value at least by the present epoch, since we observe galaxies acting as separate, bound objects today. The universe has expanded by a factor of 10^3 since decoupling[23], and so at that epoch, we require that the initial size of the density perturbation should satisfy $\delta > 10^{-3}$.

However, the temperature fluctuations in the microwave background that we discussed in Section 11.6 imply that corresponding density perturbations were only of the order of

$$\frac{\Delta T}{T} \approx \frac{\delta \rho}{\rho} \approx 10^{-5} \tag{12.16}$$

at the time of decoupling and this is too small by some two orders of magnitude. In other words, in a universe containing just baryonic matter – that is, 'ordinary' electrically charged matter such as protons, electrons etc., – there would not have been sufficient time from decoupling through to the present era for structures such as galaxies to form by gravitational attraction. We must conclude, therefore, that there must be some additional effect that we have yet to consider that can accelerate the formation of structures in the universe.

The extra ingredient comes from the non–baryonic dark matter in the universe. Since this interacts with the radiation only through gravity, it is not affected by the radiation pressure that acts on the baryons before decoupling. Consequently, the dark matter can begin to clump together once the epoch of matter–radiation equality has been attained and, as we have seen, this occurs on a much earlier timescale than the epoch of decoupling. The idea, then, is that initially the density perturbations in the distributions of baryonic and non–baryonic dark matter are equal, $\delta_{\rm baryon} \approx \delta_{\rm dm}$.

[23] Recall that the temperature of the universe at decoupling was about 3000 K, that is, about a thousand times larger than the present temperature. This implies that the scale factor of the universe has increased by this factor since $a \propto T^{-1}$.

After the epoch of matter–radiation equality, $\delta_{\rm dm}$ can grow because the universe is dominated by pressureless matter, whereas $\delta_{\rm baryon}$ is held at a fixed value due to the interaction between the (electrically charged) baryons and the radiation. Thus, by the time of decoupling, $\delta_{\rm dm} \gg \delta_{\rm baryon}$. Nevertheless, once the baryonic matter and radiation decouple, the baryons are rapidly attracted towards these higher density regions of non–baryonic dark matter. As a result, $\delta_{\rm baryon}$ can grow *more rapidly than it would do in the absence of non–baryonic dark matter* and, in principle, sufficiently quickly for galaxies to form at least by the present epoch. The clumps of dark matter that have evolved before decoupling provide the high density regions around which the baryons can cluster.

The microwave background temperature fluctuations are not enhanced by the dark matter perturbation because photons are relativistic and, consequently, are able to leak out of any small scale lumps of cold dark matter. The photons effectively travel too fast to be held back by the gravitational attraction of the dark matter. (Recall that non–baryonic dark matter only interacts with radiation through gravity because it has no electric charge). Thus, the dark matter scenario is a mechanism for enabling galaxies to form more rapidly than they otherwise would do without increasing the initial perturbation of the baryons to such an extent that the temperature fluctuations of the microwave background would exceed the observed limits.

It is important to emphasize that the structure forms because $\delta_{\rm dm}$ has been able to grow between the time when the non–baryonic dark matter dominates the dynamics of the universe and the epoch of decoupling.

The main conclusion to draw from this subsection is that the existence of galaxies can be interpreted as further evidence for non–baryonic dark matter in the universe.