13 Temperature Fluctuations in the Cosmic Microwave Background

13.1 CMB Power Spectrum

The universe was optically thick prior to the epoch of decoupling due to the scattering between free electrons and photons. This implies that we are unable to use the photons of the CMB to gain direct insight into the universe before that epoch. In effect, there is a limit to how far back in time we can 'see' with photons. On the other hand, the recombination of electrons with atomic nuclei occurred quickly (although not instantaneously). Since radiation reaches us from all directions, therefore, we can regard the photons that we observe today in the CMB as having originated from an effective spherical surface. This is known as the *surface of last scattering* – denoted LSS – and represents the projection of the decoupling epoch onto the sky. This is illustrated in Fig. (13.1).

An important property of the CMB is that it preserves its blackbody form as the universe expands after the epoch of decoupling. In principle, therefore, for each point on the sky, we could obtain the blackbody spectrum and find the best–fit temperature for the distribution. This would yield the temperature of the CMB as a function on the sky and would allow the temperature fluctuations, $\Delta T/T_0$, between any two given points on the sky to be determined.

As the distribution of the temperature fluctuations is on a spherical surface, it is standard to decompose it into spherical harmonics:

$$\frac{\Delta T(\theta,\phi)}{T_0} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\phi) \tag{13.1}$$

where Y_{lm} represent the spherical harmonics, $\{\theta,\phi\}$ are the polar coordinates on the sky and the coefficients, a_{lm} are related to the multipole moments, C_l:

$$C_l \equiv \langle |a_{lm}|^2 \rangle = \frac{1}{2l+1} \sum_m |a_{lm}|^2 \tag{13.2}$$

and $\langle \ldots \rangle$ represents an average over many realizations. The first term in the sum, $l = 0$, is the monopole and corresponds to the average temperature. The dipole ($l = 1$) is interpreted as being due to the motion of the solar system relative to the CMB rest frame and is therefore a Doppler effect. The $l \geq 2$ multipoles carry information about the intrinsic anisotropy of the CMB.

In practice, a given experiment measures the temperature difference of the CMB at two points on the sky separated by a given angle and then takes an average over all observable locations. The average is known as the correlation function and is defined by

$$C(\theta) = \langle \frac{\Delta T}{T}\left(\vec{\theta}_1\right) \frac{\Delta T}{T}\left(\vec{\theta}_2\right) \rangle \tag{13.3}$$

where $\langle \ldots \rangle$ represent the average that is performed over all points $\vec{\theta}_1$ and $\vec{\theta}_2$ separated by $\theta = \left|\vec{\theta}_1 - \vec{\theta}_2\right|$. The correlation function (13.3) is related to the multipole moments (13.2) by

$$C(\theta) = \frac{1}{4\pi} \sum_l (2l+1) C_l P_l (\cos\theta) \qquad (13.4)$$

where P_l are the Legendre polynomials. The Cosmic Background Explorer (COBE) satellite, for example, measured temperature fluctuations between points separated by approximately 10^o.

A plot of the correlation function (13.3) against angular separation, θ, would then yield the CMB power spectrum. It has become conventional to display this by plotting the combination $l(l+1)C_l$ against the multipole moment, l, where $l \propto 1/\theta$. In general, the quadrupole term, corresponding to $l = 2$, represents an angular separation of 180^o, so a given multipole, l, is related to a corresponding angular separation by

$$l \approx \frac{180}{\theta^o} \qquad (13.5)$$

13.2 Primary Sources of CMB Anisotropies

There are many physical sources that contribute to the CMB anisotropy and we can not hope to cover them all here. We will therefore focus on some of the primary ones:

1. Gravitational (Sachs–Wolfe) Perturbations: These arise because a photon moving away from a slightly overdense region on the last scattering surface (i.e. moving out of a gravitational potential well) loses energy in the process and is therefore redshifted.

2. Density (adiabatic) Perturbations: the coupling of baryonic matter and radiation can result in the compression of the radiation as well as the matter. This results in an increase in temperature.

3. Velocity (Doppler) Perturbations: The ionized plasma has a non–zero velocity. Indeed, as we discuss shortly, the density fluctuations in the matter distribution imply that it was undegoing coherent oscillations at the time of decoupling. This motion results in a Doppler shift in the observed frequency of the radiation and a corresponding shift in temperature.

13.3 Scales Explored by CMB Anisotropies

The distance to the last scattering surface is, to a very good approximation, given by the horizon distance, since decoupling occurred at a sufficiently early epoch. For a pressureless, flat universe we have seen that this is given by Eq. (9.23):

$$d_{H,0} = 3ct_0 = \frac{2c}{H_0} \qquad (13.6)$$

This corresponds to an angular separation on the sky of 180^o, so measuring the quadrupole is equivalent to measuring the temperature difference between two points at opposite ends of the surface of last scattering.

Smaller angular scales represent smaller linear scales. A scale of crucial importance is the angular scale on the last scattering surface that corresponds to the linear scale representing the horizon size at the time of decoupling. From Eq. (9.22), this is given by $d_H(t_{\rm LSS}) = 2c/H_{\rm LSS}$, where we employ a subscript 'LSS' to denote the epoch of decoupling. What we actually need here is this linear distance scaled to the present time by the factor $a_0/a_{\rm LSS} = 1 + z_{\rm LSS}$, where z is the redshift. Thus,

$$\tilde{d}_H(t_{\rm LSS}) = (1 + z_{\rm LSS})\frac{2c}{H_{\rm LSS}} = \frac{1}{\sqrt{1 + z_{\rm LSS}}}\frac{2c}{H_0} \tag{13.7}$$

where we have employed the result from Problem Set IV, Question 1(a), relating the Hubble parameter in a flat pressureless universe at a given epoch to its present–day value:

$$H(z) = H_0(1 + z)^{3/2} \tag{13.8}$$

It follows, therefore, that the angular separation on the sky corresponding to this scale is given (in degrees) by

$$\theta_{\rm LSS} = \frac{180}{\pi}\frac{\tilde{d}_H(t_{\rm LSS})}{d_{H,0}} = \frac{1}{\sqrt{1 + z_{\rm LSS}}}\frac{180}{\pi} \approx 1.7^o \tag{13.9}$$

for $z_{\rm LSS} = 1100$. This implies that the multipole corresponding to this angular scale is

$$l_{\rm LSS} \approx 100 \tag{13.10}$$

This implies that angular scales $\theta < \theta_{\rm LSS}$ correspond to linear scales that were *smaller* than the horizon size at decoupling. Conversely, angular scales $\theta > \theta_{\rm LSS}$ represent linear distance scales that were greater than the horizon size at that time. Thus, the differences in temperature between two points on the sky separated by $\theta > \theta_{\rm LSS}$ are differences between regions of the universe that would not have been in communication at the time of decoupling (since light had not had enough time to travel that distance since the big bang). In a sense, when we measure fluctuations over these scales, we are measuring the primordial perturbations that have not been affected by physical processes after the inflationary era. On the other hand, physical processes could have operated between two regions separated by $\theta < \theta_{\rm LSS}$ and we expect the temperature fluctuations over these smaller scales to reflect this. Indeed, Doppler effects are the most significant cause of temperature anisotropies over multipoles in the range $100 < l < 1000$, whereas the Sachs–Wolfe effect is the dominant contribution to the anisotropies for multipoles $l < 100$. We now consider this latter effect further.

13.4 Sachs–Wolfe Effect

The Sachs–Wolfe effect is really two competing effects arising from potential perturbations at last scattering. Firstly, an overdense region effectively *cools* the photons because they lose energy (are redshifted) as they climb out of the gravitational potential well. Secondly, an overdense region causes a time dilation effect on the surface of last scattering, because when we look at an overdense region we are looking at a hotter and therefore an effectively *younger* universe.

These effects combine to produce a fluctuation in the temperature:

$$\frac{\Delta T}{T} = \left.\frac{\Delta T}{T}\right|_{\text{grav}} + \left.\frac{\Delta T}{T}\right|_{\text{thermal}} \tag{13.11}$$

The gravitational redshift is determined by the potential well, ϕ, such that $\Delta T/T|_{\text{grav}} = \Delta\nu/\nu \equiv \phi/c^2$, where ν represents the frequency of the photons. The thermal effect is quantified by noting that the number density of photons, n_γ, at a given temperature is proportional to the density of baryons, $n_\gamma \propto \rho_B \propto a^{-3} \propto T^3$, so it follows that $3dT/T = d\rho/\rho$. Moreover, since in a pressureless universe, $a \propto t^{2/3}$ and $\rho \propto t^{-2}$, this implies that

$$\frac{d\rho}{\rho} = -2\frac{dt}{t} = 2\frac{d\nu}{\nu} = 2\frac{\phi}{c^2} \tag{13.12}$$

Noting that the gravitational and thermal contributions have opposite effects then implies that the temperature fluctuations on large angular scales are given by

$$\frac{\Delta T}{T} = \frac{\phi}{c^2} - \frac{2}{3}\frac{\phi}{c^2} = \frac{1}{3}\frac{\phi}{c^2} \tag{13.13}$$

13.5 The Predicted and Observed CMB Power Spectrum

The qualitative shape of the predicted power spectrum of the CMB is shown in Fig. (13.2). This is based on models of structure formation involving the process of gravitational instability in a dark matter inflationary universe, as outlined in Section 12. There are three regions to note: there is a flat plateau for $l < 10$, a peak located at $l \approx 200$ followed by a succession of lower peaks and then a damping tail above $l > 1000$.

The origin of the peaks in the power spectrum is that prior to decoupling, the baryonic matter in the universe was ionized and this implied that the photons could not travel freely. Instead, the matter and radiation behaved effectively as a single fluid, where the baryons provided the inertia and the photons a pressure. Due to gravitational attraction, the matter tended to collapse towards the regions of slightly higher density, but the photon pressure resisted this. This interplay between the photons and baryons set up acoustic oscillations in the fluid and the peaks and troughs represent the corresponding hot and cold regions. (The fluid is hotter in the denser regions). As we have seen, multipoles below $l < 100$ correspond to linear scales larger than the horizon at decoupling, so no peaks are seen in this region of the spectrum because physical processes had not yet had time to operate over these scales. The first peak at $l \approx 200$ corresponds to the oscillation that is just reaching its point of maximum contraction at the decoupling epoch. This angular scale of about one degree corresponds to a linear scale within the horizon distance at the time of decoupling, so the oscillation had had sufficient time to complete part of its full cycle. The other peaks may then be viewed as higher harmonics of this oscillation.

Now, it must be emphasized that the occurrence of this first peak at $l \approx 200$ is a key prediction of inflation. This multipole is special because it represents the scale corresponding to the *sound horizon* at decoupling. The sound horizon is defined as the maximum distance a sound wave could have traveled through the ionized plasma from the beginning of the universe to the time of decoupling. Quantitatively, it is the analogue of the horizon distance (9.22), but with the speed of light substituted for the speed of sound of the fluid, c_S:

$$d_S = a(t) \int_0^t \frac{c_S(t)}{a(t)} dt \tag{13.14}$$

In general, the speed of sound is defined by $c_S^2 = \partial p / \partial \rho$ and is a complicated function of time, since it is sensitive to the relative densities of matter and radiation in the early universe. However, in a radiation–dominated universe, $p = \rho c^2/3$ and $c_S = c/\sqrt{3}$. To a very crude approximation, therefore, if we take this value for the sound speed, and further assume a spatially flat universe, we deduce that the sound horizon at decoupling is smaller than the horizon distance at that time by a factor of $\sqrt{3}$. Thus, the corresponding angular scale of the sound horizon on the sky will be $\theta_S = \theta_{\rm LSS}/\sqrt{3} \approx 1^o$, represented by the multipole $l_S \approx 100 \times \sqrt{3} = 173$. This is actually quite close to the precise value determined by a full numerical calculation:

$$l_S = 200 \tag{13.15}$$

The present observational situation is best summarized by the observed power spectra measured by the Wilkinson Microwave Anisotropy Probe (WMAP). This is shown in Figs. (13.4) and (13.5). The data was made public as recently as January 2003 and further details can be found on the WMAP website at:

http://map.gsfc.nasa.gov/index.html

Note that the spectrum at low multipoles ($l < 10$) corresponds to a fluctuation in temperature of $\Delta T/T_0 \approx 10^{-5}$ as previously discussed. Suffice to say that there is now overwhelming evidence for the existence of the first peak located, as expected, at $l \approx 200$ and there is strong evidence also for the existence of the lower peaks. The curve is the best–fit theoretical model. Since, in principle, one could imagine other, non–inflationary mechanisms for generating such a power spectrum, these observational results by themselves do not prove that the universe underwent inflation. However, it is important to emphasize that these and related observations represent a crucial test that the inflationary scenario needed to pass successfully. If the peaks had not been observed, it would probably have ruled out inflation as a mechanism for generating large–scale structure in the universe.

13.6 Measuring the Density of the Universe

The precise nature of the CMB power spectrum is sensitive to many cosmological parameters, including the Ω–parameters for the cosmological constant and baryonic matter components, the Hubble parameter, etc. In particular, *the position of the first peak is sensitive to the total density of the universe and therefore its spatial curvature.* Recall that the position of this peak is related to the sound horizon at decoupling, corresponding to the maximum distance a sound wave could travel during the big bang before decoupling. This linear distance subtends an angle of the sky and, as we saw above, corresponds to about $\theta_S \approx 1^o$ for a spatially flat universe, where the total density is equal to the critical density.

Now, geometrically, a flat universe corresponds literally to a flat, Euclidean geometry and consequently light rays follow a straight line as they travel through the universe from the surface of last scattering towards our detectors. (Fig 13.8a). However, we saw in Section 5 that a closed universe corresponds to a spherical geometry. In this case, the light rays of two photons that we receive converge as we follow them back in time to the last scattering surface. Consequently, two photons separated initially by the the sound horizon at decoupling subtend a *larger* angle than in the flat universe (Fig. (13.8b)). Thus, in a closed universe, the sound horizon – and therefore the position of the first peak – will correspond to a *smaller* multipole. If the universe is closed, therefore we would expect to see the large peak located to the left of the $l = 200$ multipole. By a similar argument, for a negatively curved universe, the light rays diverge as we trace them back and subtend a smaller angle. In this case, the peak is expected to be located to the right of $l = 200$ if the density parameter is significantly less than unity. In general, it can be shown that the dependence of the position of the first peak on Ω is given by

$$l \approx \frac{200}{\sqrt{\Omega_0}} \tag{13.16}$$

Hence, by measuring the position of the first peak, we are able to measure the present density of the universe relative to the critical density and hence the curvature of the universe. The best–fit value determined from WMAP is

$$\Omega = 1.02 \pm 0.02 \tag{13.17}$$

In other words, the generic prediction of the inflationary scenario – namely, that the total density of the universe today should be very close to the critical density – is entirely consistent with the CMB data.

Such an analysis follows from purely geometrical considerations and can not by itself yield information on the different constituents of the matter content of the universe. However, when the CMB data is combined with the supernovae data that we discussed in Section 6, results such as those shown in Figs. (13.6) and (13.7) are found. For the favoured value of $h = 0.65$ (see Section 6), the best–fit to the data is for $\Omega_{\Lambda,0} = 0.7$ and $\Omega_{\mathrm{m},0} = 0.3$. This is also consistent with the independent determination of the density of dark matter deduced from galaxy cluster dynamics, as discussed in Section 10, where it was found that $\Omega_{\mathrm{m},0} \approx 0.3$.

Thus, to conclude, a concordance model of the universe has emerged, where the density is close to the critical density. Only 1% of the matter in the universe is visible, 30% is in the form of dark matter (both baryonic and non–baryonic) and the remaining 70% is comprised of a mysterious cosmological constant or dark energy.