2 The Cosmological Equations

2.1 The Friedmann Equation

The equations that describe the evolution of the universe as a whole are derived from Einstein's theory of general relativity. There are ten of these equations and they are all non–linear, partial differential equations. Solving them in any form of generality is next to impossible.

However, under the assumption of the Cosmological Principle, these equations simplify to a few ordinary differential equations. One of these is known as the *Friedmann equation* after the Russian scientist Alexander Friedmann who first derived it in 1922. This equation determines how the universe expands or contracts depending on the properties of the matter contained within it. Such a simplification is possible because the Cosmological Principle implies that the universe should be homogeneous on sufficiently large scales. In other words, at a given instant in time, physical quantities are constant throughout all of space and therefore all spatial derivatives vanish. Thus, we need only consider time derivatives.

Fortunately for us, it is not necessary to understand Einstein's theory in order to derive the Friedmann equation. It can be derived purely from Newtonian gravity, although you should be aware at the outset that such a derivation is not rigorous. Remarkably, however, this approach yields an identical result to that derived from a full treatment based on Einstein's theory of general relativity. We now proceed to derive the Friedmann equation.

Recall that the force, F, between two objects, with masses M and m separated by a distance, r, is given by

$$F = \frac{GMm}{r^2} \tag{2.1}$$

The gravitational energy associated with this force is given by

$$V = \int dr F(r) = -\frac{GMm}{r} \tag{2.2}$$

Now, consider a uniform medium that has a constant mass density, ρ. Consider a particle of mass m that is immersed in this medium and located a distance r from its centre, as shown in Fig. (2.1). To proceed, we now quote without proof the following theorem:

- *The material a distance greater that r from the centre exerts no force on the particle. Material a distance less than r exerts the same force as if all that material were concentrated at the centre.*

This is known as Newton's 'iron ball' theorem. It implies that we can ignore the effects of the matter located at distances greater that r. Only the material within the spherical shell of radius r influences the particle[4]. The mass of the material in this

[4]The reason why a Newtonian description works in the derivation of the Friedmann equation

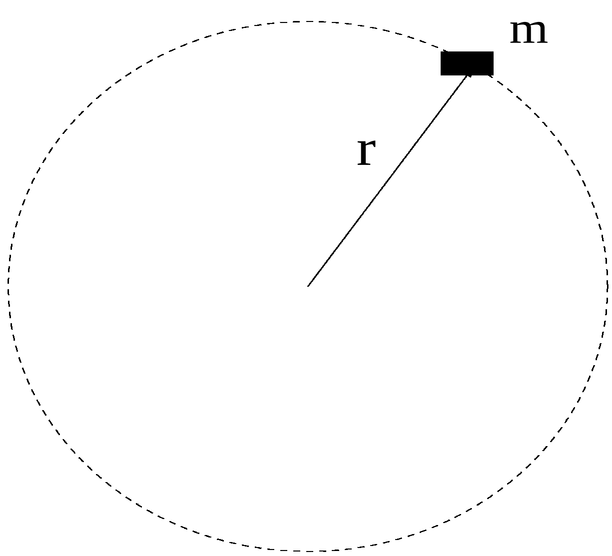

Fig. (2.1). Only the matter contained within the sphere of radius r exerts a gravitational force on a particle located a distance r from the centre.

shell is $M = 4\pi\rho r^3/3$ and the force acting on the particle is therefore

$$F = \frac{4\pi G\rho r m}{3} \tag{2.3}$$

This implies that the potential energy of the particle is

$$V = -\frac{4\pi G\rho m r^2}{3} \tag{2.4}$$

Suppose, now, that the medium is expanding and that the particle is moving along with it. The radius of the spherical shell changes at a rate[5] $\dot{r} = dr/dt$ and this corresponds to the velocity of the particle, $v = dr/dt$. The kinetic energy of the particle is therefore

$$T = \frac{1}{2}m\dot{r}^2 \tag{2.5}$$

and the total energy of the particle,

$$U = T + V = \frac{1}{2}m\dot{r}^2 - \frac{4\pi G\rho r^2 m}{3} = \text{constant} \tag{2.6}$$

is *conserved.*

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is that in an homogeneous universe, we can consider the volume of the sphere in Fig. (2.1) to be arbitrarily small and therefore sufficiently small that general relativistic effects are unimportant.

[5]Throughout this course an overdot denotes differentiation with respect to time, $\dot{y} = dy/dt$.

The homogeneity of the universe implies that a similar argument can be applied to a second particle located elsewhere in the universe. The question we must now address is how can the separation between these two particles be quantified as the universe expands?

It is helpful to think about the expansion of the universe in terms of an inflating balloon. In this analogy, space in the universe is represented by the elastic of the balloon. Consider two ants on the surface of the balloon. These ants are only aware of the two dimensions (forwards-backwards and left-right) associated with the balloon's elastic and have no concept of the third spatial dimension (up-down). The expansion of the universe is then interpreted as a *stretching* of the elastic of the balloon as it is inflated. Thus, the two ants appear to move away from each other, but actually it is the space between them that is being stretched.

If we were to then draw a grid on the surface of the balloon to determine the coordinates of the two ants, this grid would also become stretched during the expansion, as shown in Fig. (2.2). Coordinates defined in terms of such a grid system are called *comoving coordinates* because they literally move with the cosmic expansion. The comoving distance, denoted by $\vec{x}$, between the two ants remains constant, whereas the physical distance, $\vec{r}(t)$, increases as

$$\vec{r} = a(t)\vec{x} \tag{2.7}$$

where $a = a(t)$ measures the increase in the separation between the two ants.

A similar system of comoving coordinates may be applied to the cosmic expansion. Because the expansion is uniform, the grid retains its structure during the expansion. The function $a(t)$ then determines by how much space is stretched and therefore measures the size of the universe. It is referred to as the *scale factor of the universe.*

Hence, substituting Eq. (2.7) into Eq. (2.6) implies that

$$U = \frac{1}{2}mx^2\dot{a}^2 - \frac{4\pi G}{3}\rho a^2 x^2 m \tag{2.8}$$

where $x = |\vec{x}|$ is constant by definition. Rearranging, we find that

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{2.9}$$

where we have defined the constant[6]

$$k = -\frac{2U}{mc^2x^2} \tag{2.10}$$

Note that k has dimensions of $[\text{length}]^{-2}$.

[6] Many textbooks omit the factor of c^2 in the second term on the right hand side of Eq. (2.9). This is formally equivalent to setting $c = 1$. Here we keep this factor.

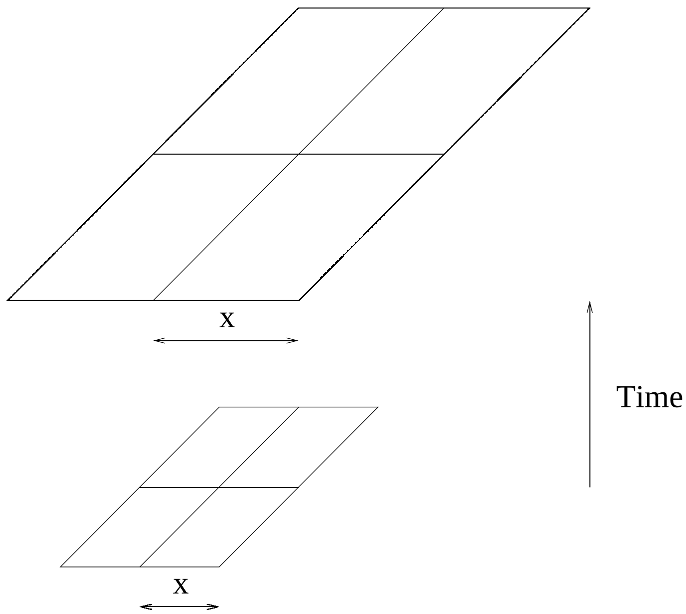

Fig. (2.2). Comoving coordinates move with the expansion of the universe. By definition, an observer moving with the expansion remains fixed relative to the comoving coordinates, in the sense that his location on the grid of comoving coordinates does not change.

Eq. (2.9) is the *Friedmann Equation*. Friedmann's equation is central to any discussion on cosmology because it tells us how the universe expands once we know the density, ρ, of the matter contained within it. It is a first–order, ordinary differential equation, albeit a non–linear one. If we knew how the density depended on the size of the universe, we could in principle solve Eq. (2.9) and determine how the scale factor varies with time. How then does the density vary?

2.2 The Conservation Equation

The second equation that we need in order to determine the dynamics of the universe is known as the *conservation equation*. This equation determines how the density of matter in the universe changes as the universe expands. It follows as a direct consequence of the first law of thermodynamics.

Consider once more the sphere of matter in Fig. (2.1) with density, ρ, pressure, p, and radius, a. The second law states that if the system undergoes an incremental change in volume, dV, the corresponding change in entropy, dS, is given by

$$dE + pdV = TdS \tag{2.11}$$

where dE represents the change in energy in the matter due to its change in volume and pdV then quantifies the work that is done by the matter in accomplishing this task.

In a homogeneous universe, all physical quantities are independent of spatial position, so there can be no spatial derivatives. In particular, there can be no temperature gradients and, consequently, no flow of heat into or out of the sphere, i.e., the total energy of the sphere remains constant. Thus, the loss in energy caused by the expansion is precisely balanced by the work done in increasing the volume, $dE = -pdV$. It then follows from the second law (2.11) that $dS = 0$ (the process is said to be reversible). Consequently, the rate of change of energy of the matter as the sphere expands is given by

$$\frac{dE}{dt} + p\frac{dV}{dt} = 0 \tag{2.12}$$

In general, the effective mass of material in a sphere of radius a is given in terms of its density by $M = 4\pi\rho a^3/3$. This mass may be associated directly with an energy by virtue of Einstein's famous equation, $E = Mc^2$, i.e.,

$$E = Mc^2 = \frac{4\pi}{3}\rho a^3 c^2 \tag{2.13}$$

The change in this energy, dE, during a time dt is then determined by differentiating (2.13) with respect to time[7]:

$$\frac{dE}{dt} = \frac{4\pi}{3}a^3c^2\frac{d\rho}{dt} + 4\pi a^2\rho c^2\frac{da}{dt} \tag{2.14}$$

and the corresponding change in volume is given by

$$\frac{dV}{dt} = 4\pi a^2\frac{da}{dt} \tag{2.15}$$

Substituting Eqs. (2.14) and (2.15) into Eq. (2.12) and rearranging then leads to the *conservation equation*:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{2.16}$$

It is important to emphasize that there can be no pressure forces in a homogeneous universe because such effects can only be generated by a pressure gradient. Consequently, pressure does not provide a force that causes the universe to expand. Rather, its contribution is entirely through the work done during the expansion.

2.3 The Equation of State

Friedmann's equation tells us how the expansion of the universe is determined by the density of matter contained within it and the conservation equation tells us how the density of matter changes as the universe expands. This is not quite the full story though. We still need to know what kind of matter is present in the universe in order to solve the equations. In other words, we need to determine how the pressure of the

[7]We must employ the chain rule here since the density also changes as the sphere expands, i.e., $\rho = \rho(t)$.

matter is related to its density. This is usually done simply by specifying the pressure directly as a function of density:

$$p = p(\rho) \tag{2.17}$$

and this relation is known as the *equation of state*. For all the situations considered in this course, the pressure varies in direct proportion to the density. Thus, we may write

$$p = (\gamma - 1)\rho c^2 \tag{2.18}$$

where γ is a constant with specific numerical values for specific types of matter and radiation.

In general, the universe contains mixtures of different types of matter, such as galaxies, radiation, etc. If these components are uncoupled, then each component, i, satisfies its own equation of state:

$$p_i = (\gamma_i - 1)\rho_i c^2 \tag{2.19}$$

and separately satisfies the conservation equation, i.e.,

$$\dot{\rho}_i + 3\frac{\dot{a}}{a}\left(\rho_i + \frac{p_i}{c^2}\right) = 0 \tag{2.20}$$

The total pressure and density are then given by $p_{\rm total} = \sum_i p_i$ and $\rho_{\rm total} = \sum_i \rho_i$.

2.4 The Acceleration Equation

There is an alternative way of writing the cosmological equations that often proves useful. Differentiating the Friedmann equation (2.9) with respect to time implies

$$2\frac{\dot{a}\ddot{a}}{a^2} - 2\frac{\dot{a}^3}{a^3} = \frac{8\pi G}{3}\dot{\rho} + 2kc^2\frac{\dot{a}}{a^3} \tag{2.21}$$

and substituting the conservation equation (2.16) into the right hand side of this equation yields

$$2\frac{\dot{a}\ddot{a}}{a^2} - 2\frac{\dot{a}^3}{a^3} = -8\pi G\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) + 2kc^2\frac{\dot{a}}{a^3} \tag{2.22}$$

Cancelling the factor of $2\dot{a}/a$ and substituting in the Friedmann equation (2.9) then implies that

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{2.23}$$

This is known as the *acceleration equation*, since it depends on the second derivative of the scale factor. (In some sense, it may be viewed as the equivalent of $F = ma$ for the universe). Notice how any dependence on the constant k that appears in the Friedmann equation has dropped out of Eq. (2.23). One immediate consequence of Eq. (2.23) is that $\ddot{a}$ is always negative in a universe in which the pressure and density

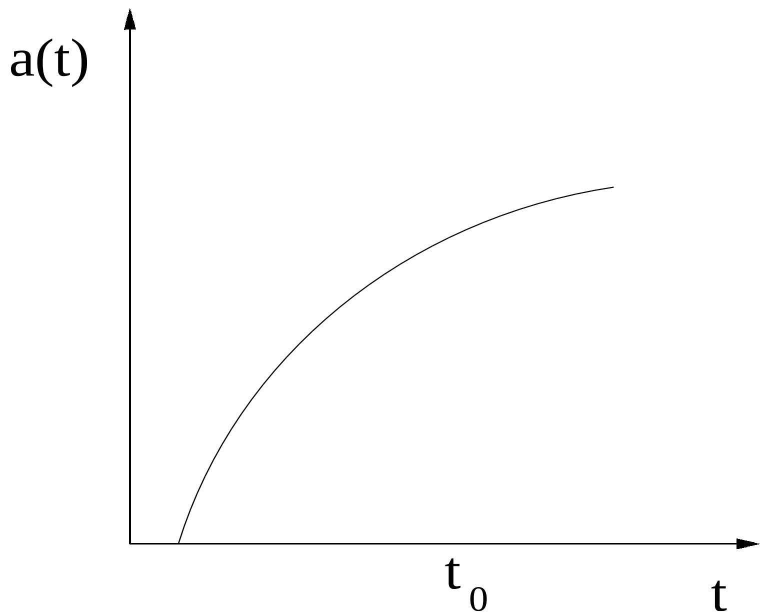

Fig. (2.3). In a universe where the density and pressure satisfy $\rho > 0$ and $p \geq 0$, $\ddot{a}$ is always negative. Hence, the gradient of the curve $a(t)$ continuously decreases with time. Conversely, as we go back in time, it increases and this implies that $a(t)$ must necessarily cross the time axis at some finite time in the past. The intersection can be interpreted as the origin of the universe.

are non–negative, i.e., $p \geq 0$ and $\rho > 0$. This implies that $\dot{a}$ must be either positive or negative, i.e., the universe must be either expanding or contracting. (Since Hubble's constant is positive and $a > 0$ by definition, the universe is expanding). Crucially, *the universe can not be static*[8].

Moreover, the minus sign of the right hand side of this equation implies that the expansion of the universe is slowing down with time. This is precisely what we would expect since the attractive nature of gravity resists the universe's efforts to expand. This behaviour also implies that as we go back in time, $\dot{a}$ must increase and so the curve $a(t)$ must concave downwards towards the t–axis. As a result, there must be a time in the finite past when the scale factor vanishes (see Fig. (2.3)). This is a point of zero volume and infinite density – otherwise known as the big bang. It is important to appreciate that we arrive at such a conclusion just by requiring that the rather weak conditions $\rho > 0$ and $p \geq 0$ are satisfied by the matter.

2.5 Hubble's Law Revisited

Given the Friedmann equation (2.9), the constant of proportionality in Hubble's law, Eq. (1.8), now acquires a physical interpretation. Recall that Hubble's law implies that as we look out in a direction $\vec{r}$, the velocity of recession of a galaxy is given by

[8] It is possible for $\dot{a}$ to vanish instantaneously, but this just corresponds to the expansion changing over to a contraction.

$\vec{v} = H_0 \vec{r}$. However, the velocity may also be written as

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{|\dot{\vec{r}}|}{|\vec{r}|}\vec{r} = \frac{\dot{a}}{a}\vec{r} \tag{2.24}$$

where we have substituted the definition (2.7) for $\vec{r}$ and remembered that by definition $\vec{x}$ is independent of time. Thus, comparison of Eq. (2.24) with Hubble's law (1.8) implies that the constant of proportionality may be identified with the ratio

$$H_0 = \frac{\dot{a}_0}{a_0} = \left(\frac{d \ln a}{dt}\right)_0 \tag{2.25}$$

where subscript '0' denotes that quantities are to be evaluated at the present epoch. The Hubble constant is therefore the logarithmic derivative of the scale factor evaluated at the present era.

In general, the ratio $\dot{a}/a$ is known as the *Hubble parameter* and is denoted by H:

$$H \equiv \frac{\dot{a}}{a} \tag{2.26}$$

The Hubble parameter varies with time as the universe evolves. The cosmological Friedmann, conservation and acceleration equations can then be written in a more compact form in terms of the Hubble parameter:

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{2.27}$$

$$\dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0 \tag{2.28}$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{2.29}$$

We now have all we need in the form of Eqs. (2.27), (2.28) and (2.29), together with the equation of state (2.17), to quantitatively discuss the entire history of the universe. This involves solving this set of equations and is the topic of the next Section.