3 Solving the Cosmological Equations

3.1 A Pressureless Universe

An obvious starting point is to describe the present expansion of the universe. To do this, we need to specify the equation of state at the present epoch. Galaxies behave as massive, gravitationally–bound entities. Since the typical separation between galaxies (1 Mpc) is much larger than the average size of a galaxy (10^{-2} Mpc), collisions between galaxies are rare. We may therefore view galaxies as individual 'particles' that are massive but have no internal structure and do not collide with each other. This implies that there is no pressure between them, so it is reasonable to assume that the matter in the universe today is pressureless:

$$p = 0 \tag{3.1}$$

This is equivalent to choosing $\gamma = 1$ in the equation of state (2.17).

In this case, the conservation equation simplifies to

$$\frac{d\rho}{dt} + 3\left(\frac{1}{a}\frac{da}{dt}\right)\rho = 0 \tag{3.2}$$

and to solve this equation, we make use of the identity

$$\frac{1}{a^3}\frac{d}{dt}\left(\rho a^3\right) = \frac{d\rho}{dt} + \frac{3}{a}\frac{da}{dt}\rho \tag{3.3}$$

We deduce therefore that

$$\frac{d}{dt}\left(\rho a^3\right) = 0 \quad \Longrightarrow \quad \rho a^3 = \text{constant} \tag{3.4}$$

In other words, the density of matter falls inversely with the volume of the universe, as expected.

It is conventional to express the constant of proportionality in terms of the values taken by the parameters at the present epoch. These are denoted by a subscript '0', so for example, $\rho_0 = \rho(t_0)$, where t_0 denotes the present time. Thus,

$$\rho(t) = \rho_0 \left(\frac{a_0}{a}\right)^3 \tag{3.5}$$

We can now substitute this solution into the Friedmann equation (2.27) to determine how the scale factor varies with time. The algebraically simplest case to consider is for $k = 0$, in which case

$$a\left(\frac{da}{dt}\right)^2 = \frac{8\pi G \rho_0 a_0^3}{3} = \text{constant} \tag{3.6}$$

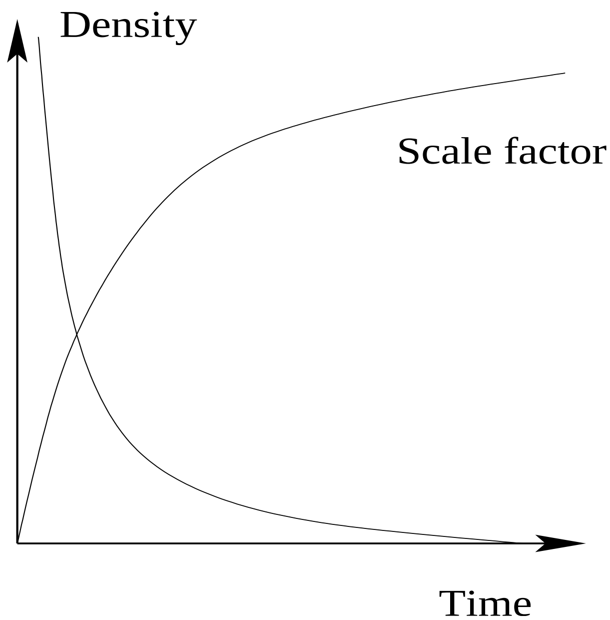

Fig. (3.1). Qualitative behaviour of the scale factor and density of matter in a $k = 0$ universe containing pressureless matter.

The simplest way to proceed is to 'guess' that the solution is of the form

$$a = \kappa t^n \tag{3.7}$$

where n and κ are constants to be determined. Differentiating Eq. (3.7) and substituting into Eq. (3.6) then implies that

$$a\left(\frac{da}{dt}\right)^2 = \kappa^3 n^2 t^{3n-2} \tag{3.8}$$

and since the right hand side of this equation must be independent of time, we must choose $n = 2/3$. The remaining constant κ is then determined by noting that $a = a_0$ when $t = t_0$. Thus, the desired solution is

$$a = a_0 \left(\frac{t}{t_0}\right)^{2/3} \tag{3.9}$$

and substituting Eq. (3.9) back into Eq. (3.5) implies that the density of matter falls with time as

$$\rho(t) = \rho_0 \left(\frac{t_0}{t}\right)^2 \tag{3.10}$$

The solutions are sketched in Fig. (3.1). The universe expands forever, with the galaxies gradually moving further and further away from each other. As we go sufficiently far back in time ($t \to 0$), we find that $a \to 0$ and $\rho \to \infty$. This indicates that the universe started out with infinite density and zero volume – a big bang.

We should be careful about drawing any firm conclusions about the origin of the universe from this solution, however, because our assumption that the galaxies rarely collide certainly breaks down at smaller volumes (earlier times). As we shall see later, our approximation that the pressure of the matter vanishes remains valid once the universe is about 10,000 years old.

3.2 A Universe Dominated by Radiation

Before proceeding we need to clarify a distinction between mass density, ρ, and energy density, often denoted by ϵ. The effects of radiation are usually quantified in terms of energy density, i.e., the energy per unit volume. However, the Friedmann and conservation equations are equations in the effective mass density. Hence, we must convert to mass density when determining cosmological behaviour. This is simply done with Einstein's equation $E = mc^2$, so dividing by the volume we have

$$\epsilon = \rho c^2 \tag{3.11}$$

We saw in Section 1.5 that the microwave background radiation has a thermal, or blackbody, spectrum. It can be shown that the energy density, ϵ, of blackbody radiation in a frequency interval df is given in terms of its frequency, f, by

$$\epsilon(f)df = \frac{8\pi h}{c^3}\frac{f^3}{\exp(hf/k_BT) - 1}df \tag{3.12}$$

where k_B is Boltzmann's constant. The qualitative form of this function is shown in Fig. 8 handed out in the first lecture. In general, the blackbody spectrum is peaked at the frequency $f_{\text{peak}} \approx 2.8k_BT/h$ corresponding to an energy $E_{\text{peak}} = hf \approx 2.8k_BT$. The mean energy of radiation (technically the photons) is $\langle E\rangle \approx 3k_BT$.

The total energy density of the blackbody radiation is deduced by integrating Eq. (3.12) over all frequencies:

$$\epsilon_{\text{rad}} = \int_0^\infty df \epsilon(f) \tag{3.13}$$

By defining the new variable $x \equiv hf/k_BT$, we may write this as

$$\epsilon_{\text{rad}} = \frac{8\pi k_B^4}{h^3c^3}T^4 \times \int_0^\infty dx \frac{x^3}{e^x - 1} \tag{3.14}$$

The integral can be evaluated and has the numerical value $\pi^4/15$ (see Appendix B for its derivation in terms of the Riemann zeta function). It follows, therefore, that the energy density of the radiation scales as the *fourth* power of its temperature:

$$\epsilon_{\text{rad}} = \alpha T^4 \tag{3.15}$$

where the constant of proportionality

$$\alpha = \frac{\pi^2 k_B^4}{15\hbar^3c^3} = 7.565 \times 10^{-16}\,\text{J}\,\text{m}^{-3}\,\text{K}^{-4} \tag{3.16}$$

is known as the radiation constant. The observed temperature of the CMB, $T_0 = 2.728\,\mathrm{K}$, corresponds to an energy density of

$$\epsilon_{\rm rad} = 4.19 \times 10^{-14}\,\mathrm{J\,m^{-3}} \tag{3.17}$$

and an effective mass density of

$$\rho_{\rm rad} = \frac{\epsilon_{\rm rad}}{c^2} = 4.66 \times 10^{-31}\,\mathrm{kg\,m^{-3}} \tag{3.18}$$

where we make use of the fact that $1\,\mathrm{J} = 1\,\mathrm{kg\,m^2 sec^{-2}}$ (see the table in Appendix A).

To understand how a universe dominated by radiation expands, we need to derive the equation of state for radiation. This can be achieved by returning to the second law of thermodynamics, Eq. (2.11). This can be rewritten in the form

$$TdS = d\left[(\rho c^2 + p)V\right] - V\,dp \tag{3.19}$$

where we write $dE = d(\rho c^2 V)$. We are looking for a relationship between the temperature and pressure of the radiation. This can be derived by viewing the entropy as a function of both the volume and the density (or equivalently the temperature) of the system, i.e., $S = S(V, \rho)$. The chain rule then implies that

$$dS = \left(\frac{\partial S}{\partial V}\right)_T dV + \left(\frac{\partial S}{\partial T}\right)_V dT \tag{3.20}$$

where a subscript $(\ldots)_X$ denotes that the derivative is performed with the variable X held constant.

Setting $d\rho = 0$ (constant density) and dividing by dV implies

$$\frac{\partial S}{\partial V} = \frac{\rho c^2 + p}{T} \tag{3.21}$$

On the other hand, fixing the volume, $dV = 0$, and dividing by dT implies

$$\frac{\partial S}{\partial T} = \frac{V}{T}\frac{\partial(\rho c^2)}{\partial T} \tag{3.22}$$

The integrability condition for partial derivatives is (the statement that partial derivatives commute):

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T} \tag{3.23}$$

Differentiating Eq. (3.21) with respect to temperature and Eq. (3.22) with respect to volume and substituting into Eq. (3.23) then implies that

$$\frac{\partial}{\partial T}\left(\frac{\rho c^2 + p}{T}\right) = \frac{1}{T}\frac{\partial(\rho c^2)}{\partial T} \tag{3.24}$$

This expression simplifies with the help of the chain rule to

$$\frac{1}{T}\frac{\partial p}{\partial T} + (\rho c^2 + p)\frac{\partial T^{-1}}{\partial T} = 0 \tag{3.25}$$

and rearranging implies that

$$\frac{dp}{dT} = \frac{1}{T}(\rho c^2 + p) \tag{3.26}$$

where we convert back to an ordinary derivative since in an homogeneous universe, the pressure is a function only of temperature.

We may employ Eq. (3.26) to deduce how the pressure of blackbody radiation is related to its energy density, $\epsilon_{\rm rad} = \rho_{\rm rad} c^2$. Substituting Eq. (3.15) into Eq. (3.26) implies that

$$\frac{dp}{dT} = \alpha T^3 + \frac{p}{T} \tag{3.27}$$

and this equation is readily solved when the pressure is given by $p = \alpha T^4/3$, as can be verified by direct substitution. Hence, the pressure of the radiation is related to its density by

$$p = \frac{1}{3}\rho c^2 \tag{3.28}$$

This is the equation of state for a universe dominated by radiation. Comparison with Eq. (2.17) implies that $\gamma = 4/3$.

The Friedmann and conservation equations (2.27) and (2.28) may now be solved, at least when $k = 0$. The method is similar to that adopted in Section 3.1 for a pressureless universe (see Problem Sheet I). The result is

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{1/2} \tag{3.29}$$

$$\rho(t) = \rho_0 \left(\frac{a_0}{a}\right)^4 = \rho_0 \left(\frac{t_0}{t}\right)^2 \tag{3.30}$$

As for pressureless matter, the density falls as t^{-2} in a universe dominated by radiation, although the expansion of such a universe is slower. This is because the extra pressure due to the radiation introduces an effective energy density that can in turn be viewed in terms of a mass contribution via $E = mc^2$. Hence, the gravitational pull of the matter is stronger and resists the expansion. Finally, note that the scale factor vanishes and that the density and temperature diverge as $t \to 0$.

3.3 Redshift Revisited

Why does the density of radiation fall as the *fourth* power of the scale factor? Three of the powers are due to the volume increase. The extra power is due to the *stretching* of the radiation's wavelength during the cosmic expansion.

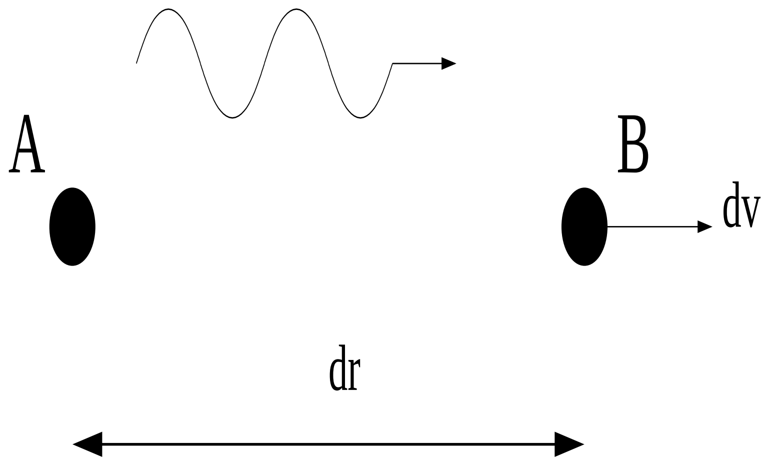

Fig. (3.2). A pulse of radiation travels a distance dr between two observers A and B with a separation speed dv.

To see this, consider two hypothetical observers A and B separated by a distance dr with a recession velocity dv. (Fig. (3.2)). Hubble's law (1.3) implies that

$$dv = H dr = \frac{\dot{a}}{a} dr \tag{3.31}$$

Suppose that A sends B a pulse of light. Due to their relative motion, the wavelength, λ, of the light received by B is greater than that emitted by A. The difference, $d\lambda = \lambda_{\rm received} - \lambda_{\rm emitted}$, is determined by Doppler's law (1.1):

$$\frac{d\lambda}{\lambda_{\rm emitted}} = \frac{dv}{c} = \frac{\dot{a}}{a} \frac{dr}{c} \tag{3.32}$$

The time it takes the light to travel this incremental distance is $dt = dr/c$, so

$$\frac{d\lambda}{\lambda_{\rm emitted}} = \frac{\dot{a}}{a} dt = \frac{da}{a} \tag{3.33}$$

and integrating this expression implies that

$$\ln \lambda = \ln a + \text{constant} \tag{3.34}$$

or, equivalently,

$$\lambda \propto a \tag{3.35}$$

The energy of the radiation therefore scales as $E \propto \lambda^{-1} \propto a^{-1}$ and, consequently, the energy density scales as $\rho_{\rm rad} \propto a^{-4}$.

We may employ this property to derive a crucially important relationship between the scale factor of the universe and redshift. Eq. (3.35) implies that the ratio of the wavelength, $\lambda_{\rm em}$, of light emitted by a distant galaxy to that observed by us, $\lambda_{\rm obs}$, is related to the corresponding scale factors by

$$\frac{\lambda_{\rm obs}}{\lambda_{\rm em}} = \frac{a_{\rm obs}}{a_{\rm em}} \tag{3.36}$$

However, this ratio is also determined in terms of the redshift from the definition (1.2) such that

$$1 + z = \frac{a_0}{a(t)} \tag{3.37}$$

Eq. (3.37) is important because it relates redshift directly to the size of the universe. So, when we say that a galaxy has a redshift of one, for example, this implies that the light that we receive from it today was emitted when the universe was one half its present size.

3.4 A Universe containing a Mixture of Matter and Radiation

Note: In the lectures, we did not cover the somewhat detailed derivation of the time of equality, Eq. (3.49), presented here. I have included it in the text, however, in case you are interested in looking at this topic further.

The universe today is comprised of a mixture of matter (with zero pressure) and radiation. Galaxies appear to be electrically neutral and so it is reasonable to assume that the matter and radiation are effectively uncoupled, in the sense that they do not interact through electromagnetism. Thus, both types of matter satisfy the conservation equation (with the appropriate equation of state) and the total density is given by $\rho_{\rm total} = \rho_r + \rho_m$, where $\rho_r \propto a^{-4}$ and $\rho_m \propto a^{-3}$ for radiation and matter, respectively. The ratio of the radiation and matter densities scales as $\rho_r/\rho_m \propto a^{-1}$ and falls as the universe expands. We therefore expect the matter to dominate the radiation at sufficiently late times (assuming implicitly that the universe continues to expand indefinitely). On the other hand, the radiation will dominate at earlier times.

In order to determine when the matter does come to dominate, we need to solve the the Friedmann equation for a universe comprised of such a mixture of matter and radiation. Fortunately, this can be done at least when $k = 0$. In this case, the Friedmann equation is given by

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\left[\frac{\rho_{m0}a_0^3}{a^3} + \frac{\rho_{r0}a_0^4}{a^4}\right] \tag{3.38}$$

where we have normalized the densities in terms of present–day values, as usual. To proceed, it is algebraically convenient to define two constants

$$\sigma_m \equiv \frac{8\pi G\rho_{m0}a_0^3}{3}, \qquad \sigma_r \equiv \frac{8\pi G\rho_{r0}a_0^4}{3} \tag{3.39}$$

Taking the positive square root of the Friedmann equation (3.38) (since we are interested in expanding universes) then implies that

$$a\frac{da}{dt} = \left[\sigma_m da + \sigma_r\right]^{1/2} \tag{3.40}$$

Eq. (3.40) can be solved by separation of variables and integrating by parts. This yields the solution in terms of $t(a)$, i.e., in terms of time as a function of scale factor. Unfortunately, the solution can not be inverted to yield $a = a(t)$ and is not particularly illuminating. However, a parametric solution can be found by defining a new dependent variable

$$\eta \equiv \int \frac{dt}{a} \quad \Longrightarrow \quad d\eta = \frac{dt}{a} \quad \Longrightarrow \quad a\frac{d}{dt} = \frac{d}{d\eta} \tag{3.41}$$

Note that $dt > 0 \Longleftrightarrow d\eta > 0$ and consequently increasing t corresponds to increasing η. This implies that the variable η make a good time variable. Substituting Eq. (3.41) into (3.40) and separating the variables then yields

$$\int_0^a da\left[\sigma_r + \sigma_m a\right]^{-1/2} = \int_0^\eta d\eta \tag{3.42}$$

Here we have chosen the limits of integration so that the scale factor vanishes at $\eta = 0$, since the discussion in Section 2.4 has shown us that the scale factor must vanish at some finite time in the past.

Evaluating the integral (3.42) yields

$$\frac{2}{\sigma_m}\left[\sigma_r + \sigma_m a\right]^{1/2} - \frac{2\sqrt{\sigma_r}}{\sigma_m} = \eta \tag{3.43}$$

and rearranging implies that

$$a(\eta) = \frac{\sigma_m}{4}\eta^2 + \sqrt{\sigma_r}\eta \tag{3.44}$$

The dependence of time, t, on η is now deduced by substituting Eq. (3.44) into Eq. (3.41):

$$\int_0^t dt = \int_0^\eta d\eta\, a(\eta) \tag{3.45}$$

where we have chosen the limits so that the origin of the universe occurs at $t = 0$. We find that

$$t(\eta) = \frac{\sigma_m}{12}\eta^3 + \frac{\sqrt{\sigma_r}}{2}\eta^2 \tag{3.46}$$

Eqs. (3.44) and (3.46) represent a parametric solution describing the expansion of a universe filled with matter and radiation with $k = 0$. The asymptotic limits of the solution at early ($t \to 0$) and late ($t \to \infty$) times can be deduced. At early times, $t \propto \eta^2 \to 0$ and $a \propto \eta \propto t^{1/2}$. This is the behaviour given in Eq. (3.29) for a universe containing just radiation and implies that the radiation dominates over the matter during the early history of the universe. At late times, $t \propto \eta^3 \to \infty$, and $a \propto \eta^2 \propto t^{2/3}$. This is the expansion for a universe comprised of only pressureless matter (see Eq. (3.9)). Thus, at sufficiently late times, the matter will dominate over the radiation, whereas the radiation dominates at very early times ($a \to 0$), as anticipated above.

The key question to address at this stage, therefore, is when does the transition from radiation domination to matter domination occur? This epoch is referred to as the *epoch of matter–radiation equality*, and is denoted $t_{\rm eq}$. We may now employ the solution (3.44) and (3.46) to estimate $t_{\rm eq}$.

Matter–radiation equality occurs when the densities of the matter and radiation are equal. Thus, by equating the two terms on the right–hand side of the Friedmann equation (3.38), the scale factor at this time is given in terms of its present–day value by

$$a_{\rm eq} = \frac{\rho_{r0}}{\rho_{m0}} a_0 = \frac{\sigma_r}{\sigma_m} \tag{3.47}$$

where $a_{\rm eq} = a(t_{\rm eq})$. Given Eq. (3.47), we can substitute into Eq. (3.44) to determine $\eta_{\rm eq}$:

$$\eta_{\rm eq} = 2\left(\sqrt{2} - 1\right) \frac{\sqrt{\sigma_r}}{\sigma_m} \tag{3.48}$$

Then, substituting Eq. (3.48) into the solution (3.46) allows us to determine when matter–radiation equality occurs. We find, after a little algebra, that

$$t_{\rm eq} = \frac{2\sqrt{2}(\sqrt{2}-1)}{3} \frac{\sigma_r^{3/2}}{\sigma_m^2} \tag{3.49}$$

or, in other words, that

$$t_{\rm eq} = \frac{2\sqrt{2}\left(\sqrt{2}-1\right)}{3} \left(\frac{8\pi G \rho_{m0}}{3}\right)^{-1/2} \left(\frac{\rho_{r0}}{\rho_{m0}}\right)^{3/2} \tag{3.50}$$

where we have also employed Eq. (3.39). Note that this is determined entirely in terms of the present–day densities of radiation and matter.

Let us now estimate the numerical value of $t_{\rm eq}$. The current mass density of radiation is given by Eq. (3.18). We can estimate the current density of matter in the universe by noting that its size is $\mathcal{O}(10^{26})\,{\rm m}$ and if there are 10^{11} galaxies each with 10^{11} stars, its mass is $\mathcal{O}(10^{22}) M_\odot \approx 10^{52}\,{\rm kg}$. Thus, $\rho_{m0} \approx 10^{-26}\,{\rm kg\,m^{-3}}$. This is much larger than that of the radiation, $\rho_{m0} \gg \rho_{r0}$, and we conclude that the universe is presently dominated by the matter and not by the radiation. Moreover, in view of this, the second term on the right hand side of Eq. (3.50) may be directly related to the present value of the Hubble constant:

$$t_{\rm eq} \approx \frac{2\sqrt{2}\left(\sqrt{2}-1\right)}{3} \frac{1}{H_0} \left(\frac{\rho_{r0}}{\rho_{m0}}\right)^{3/2} \approx 1.4 H_0^{-1} \times 10^{-7} \approx 2000\,{\rm yr} \tag{3.51}$$

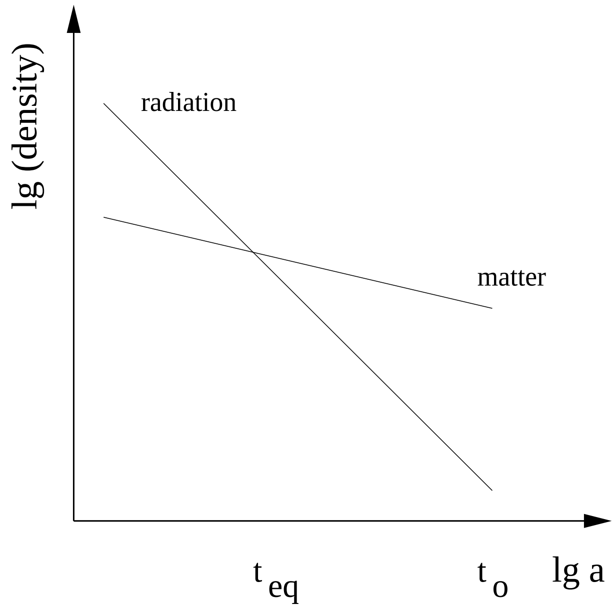

Fig. (3.3). A logarithmic plot of the redshift of matter and radiation densities. The universe has been matter dominated since $t_{\rm eq} \approx 10^4$ yr up to the present time $t_0 \approx 6.5h^{-1}$ Gyr.

It turns out that the actual figure is closer to 10,000 years for the epoch of matter–radiation equality. The slight underestimation has arisen due to our inaccuracies in estimating the relevant densities. In particular, there is also a background flux of neutrinos that behaves as a type of radiation. However, we must defer a more complete discussion of these and related questions to later Sections.

The key point to extract from Eq. (3.51) is that the matter has dominated the dynamics of the universe for most of its history. The qualitative history of the universe is sketched in Fig. (3.3). Since the universe has been dominated by pressureless matter since the universe was a few thousand years old, we can now estimate its age, at least assuming the special case $k = 0$. (We will consider cases where $k \neq 0$ later on). The solution (3.9) implies the Hubble parameter varies with time as

$$H = \frac{1}{a}\frac{da}{dt} = \frac{2}{3t} \tag{3.52}$$

and so the age of the universe is given by

$$t_0 = \frac{2}{3H_0} = 6.5h^{-1}\,\text{Gyr} \tag{3.53}$$

In other words, by measuring the Hubble parameter, we may estimate the age of the universe.

The epoch of matter–radiation equality is important, as we shall see, because it is the earliest time that structures could have started to form in the universe (see the Chapter on structure formation).