4 Physical Cosmology I

4.1 Setting the Scene

The focus of (roughly) the next quarter of the course will be on some of the parameters in cosmology that can actually be observed and the information and conclusions that can be drawn by measuring their values.

In general, many of the key characteristics of the universe are determined in terms of just three parameters. These are (i) the Hubble parameter, H_0, which determines the expansion rate of the universe; (ii) the deceleration parameter, q_0, which measures how fast the expansion of the universe is either slowing down or speeding up and; (iii) the Ω–parameter, Ω_0, which quantifies the density of matter in the universe. Once we know the values of these parameters, it is possible (in principle) to determine important quantities such as the present age of the universe as well as gain insight into its future destiny. However, as we shall see, the considerable difficulties involved in measuring these parameters mean that some uncertainties still remain.

We have already discussed the Hubble parameter in Section 1.4, and seen how it is related to the age of a pressureless universe, at least when $k = 0$. In the following subsection we introduce the so–called Ω–parameter. The value of this parameter determines whether the universe will expand forever or ultimately recollapse. We defer until Section 6 a discussion on the deceleration parameter. The deceleration parameter is closely related to the Hubble parameter and quantifies measurable deviations from Hubble's law at high redshifts. However, in order to consider such effects concretely, we first need to discuss more precisely how distances are measured in the universe. This requires some knowledge of how distances are measured in General Relativity and to proceed, we will need to introduce the concept of a spacetime metric in a cosmological context. We discuss the key features of the metric in Section 4.3. Once we have done this, we will be in a position to relate comoving distances directly to redshift – a key observable quantity, as we have already seen. This we do in Section 6.

4.2 The Density and Destiny of the Universe

4.2.1 The Ω–Parameter

The Ω–parameter is a dimensionless way of expressing how much matter is present in the universe. From the Friedmann equation (2.27), we can see that for a given value of the Hubble parameter, H, there is a certain density for which $k = 0$. This is known as the *critical density*, $\rho_c(t)$, and is defined as

$$\rho_c(t) \equiv \frac{3H^2}{8\pi G} \tag{4.1}$$

The critical density is a function of time because the Hubble parameter also varies with time.

$\rho > \rho_c$	$\Omega > 1$	$kc^2 = +1$	Closed	Recollapses
$\rho = \rho_c$	$\Omega = 1$	$kc^2 = 0$	Flat	Just Expands Forever
$\rho < \rho_c$	$\Omega < 1$	$kc^2 = -1$	Open	Expands Forever

Table (4.2.1): The relationship between the density, curvature, and destiny of the universe. The critical density, ρ_c, is defined in Eq. (4.1).

The definition of the Ω–parameter is then given in terms of the ratio of the density of matter in the universe to the critical density:

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G\rho}{3H^2} \tag{4.2}$$

We also use this notation to quantify the densities of individual matter components, e.g., for a mixture of pressureless matter and radiation, $\Omega_{\rm mat} = \rho_{\rm mat}/\rho_c$ and $\Omega_{\rm rad} = \rho_{\rm rad}/\rho_c$, respectively. The total density is then given by then sum over all the matter species, $\Omega_{\rm total} = \sum_i \Omega_i$.

The Friedmann equation (2.27) may be rewritten in terms of the Ω–parameter. Substituting the definition (4.1) into Eq. (2.27) yields

$$\frac{\rho}{\rho_c} - \frac{kc^2}{a^2H^2} = 1 \tag{4.3}$$

and substituting the definition (4.2) into Eq. (4.3) then implies that

$$\Omega - 1 = \frac{kc^2}{a^2H^2} \tag{4.4}$$

Eq. (4.4) relates the Ω–parameter directly to the constant k. Since the ratio k/a^2 has dimensions of [length]$^{-2}$, it can be given a physical interpretation in terms of the *curvature* of the universe. Thus, the term in the Friedmann equation (2.27) that contains the constant k may be referred to as the *curvature term*. In short, the curvature of the universe is determined by the sign of k. It is conventional to normalize k so that it takes one of three possible values, $kc^2 = +1, 0, -1$. (Since k only appears in the Friedmann equation in the combination k/a^2, any other non–zero value of k can be rescaled to unity simply by rescaling the scale factor).

The curvature of the universe is then positive for $k > 0$, zero for $k = 0$ and negative for $k < 0$. A positively–curved universe is similar in shape to a sphere and has finite volume. It is often referred to as a 'closed' universe. The universe is said to be 'flat' if it has zero curvature. If the curvature is negative, the universe may be viewed as a (higher–dimensional) equivalent of a horse's saddle (see Section 4.3 and Fig. (4.4)). In this case the volume is infinite and the universe is said to be 'open'. The three possibilities are summarized in Table 4.2.1.

We see from Eq. (4.4) that $\Omega = 1$ when $k = 0$. This is a very special value because if $\Omega = 1$ initially, then it stays fixed at this value for all time. In general, if $\Omega \neq 1$, it is time–dependent[9]. For an open universe ($kc^2 = -1$), $\Omega < 1$ and for a closed universe ($kc^2 = +1$), $\Omega > 1$. It is important to emphasize that since the sign of the right hand side of Eq. (4.4) is fixed by the sign of the curvature k, Ω can never pass from being greater than unity to less than unity and vice–versa, i.e., if it is less than (greater than) unity at some time, it remains so throughout the entire history of the universe.

4.2.2 The Destiny of the Universe

In short, the curvature determines the allowed range of values for the Ω–parameter. The value of the Ω–parameter today, Ω_0, also determines the future behaviour of the universe. We saw in Section 3.1 that a flat universe dominated by pressureless matter expands indefinitely, with the expansion rate progressively slowing down. Can the expansion ever be reversed into a recollapse?

For the expansion to be halted and reversed requires $\dot{a} = 0$, or equivalently, $H = 0$, for some instant of time. We see immediately from the Friedmann equation (2.27) that the right hand side is always positive if $k \leq 0$ (or equivalently if $\Omega \leq 1$) and in these cases the expansion continues indefinitely. Thus, a necessary condition for the universe to recollapse is that $k > 0$ and $\Omega > 1$.

Consider such a universe containing pressureless matter and radiation. Since the respective densities of these components fall away more rapidly than the kc^2/a^2 curvature term in the Friedmann equation, this term becomes progressively more important as the expansion of the universe proceeds. For $k > 0$, a time comes when the curvature term precisely cancels the effects of the radiation and matter densities. At this point, the expansion of the universe comes to a halt. Moreover, since the acceleration equation (2.29) implies that $\ddot{a} < 0$, this must correspond to a maximum in the scale factor. Consequently, a recollapse of the universe follows as gravity takes over. Ultimately, this recollapse ends in a 'big crunch' (see Fig. (4.1)).

The important point to emphasize is that destiny of the universe is determined by the density of matter within it and this in turn is determined by the value of Ω. So, by measuring Ω_0 today, we could deduce whether the universe will continue to expand forever, or whether it will ultimately recollapse. The options are summarized in Table 4.2.1. This is the reason why the critical density (4.1) is so called – it represents the critical density that just allows the universe to expand to infinity. If the density of the universe exceeds this value (for a given value of the Hubble parameter), it will ultimately recollapse.

4.2.3 Observed Value of Ω_0

It is important therefore to determine the present–day value of $\Omega_0 = \rho_0/\rho_c(t_0)$. The numerical value of the critical density is deduced by substituting in the present–day

[9] There is another special case where Ω = constant. This arises when the scale factor varies linearly with time, $a \propto t$, since then $\dot{a}^2 = a^2 H^2$ = constant.

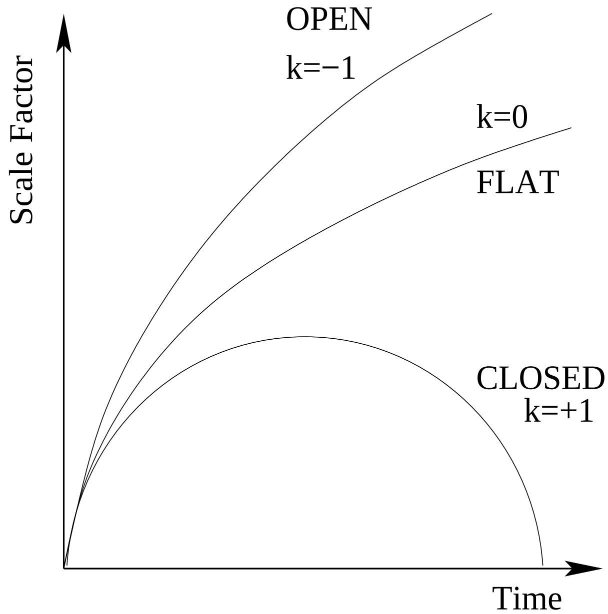

Fig. (4.1). The fate of the universe is determined by how much matter is present and therefore by the value of the Ω–parameter, or equivalently, the curvature of the universe.

value of the Hubble constant, as given by Eq. (1.5), into the definition (4.1):

$$\rho_c(t) = 1.88h^2 \times 10^{-26}\,\mathrm{kg\,m^{-3}} = 2.78h^2 \times 10^{11} M_\odot \mathrm{Mpc}^{-3} \tag{4.5}$$

Note that this is a very, very *small* value. (The density of water is $10^3\,\mathrm{kg\,m^{-3}}$).

We have previously estimated the current density of the universe to be $\rho_0 \approx 10^{-26}\,\mathrm{kg\,m^{-3}}$. Since our observations of the Hubble constant indicate that $h \approx \mathcal{O}(1)$, this implies that *the density of the universe today is comparable to the critical density and therefore that* $\Omega_0 \approx 1$. An alternative way of arriving at this estimate is to note that the typical separation between galaxies is about 1 Mpc and that the typical mass of a galaxy is about $10^{11} M_\odot$. The fact that Ω_0 is close to unity today (at least to within an order of magnitude) has profound implications for our understanding of the very early universe, as we shall see in the Section outlining the problems of the big bang model.

We must defer a more detailed discussion of the measured value of Ω_0 until the Section on dark matter in the universe. At this stage, we simply note that the favoured value for the density of matter in the universe is

$$\rho_{\mathrm{matter},0} \approx 0.3\rho_c \qquad \Longrightarrow \qquad \Omega_{\mathrm{matter},0} \approx 0.3 \tag{4.6}$$

This value is deduced from observations of the dynamics of galaxy clusters. It indicates that the density of matter is insufficient to close the universe and that the universe may be open. (See, however, the later Section on the cosmological constant).

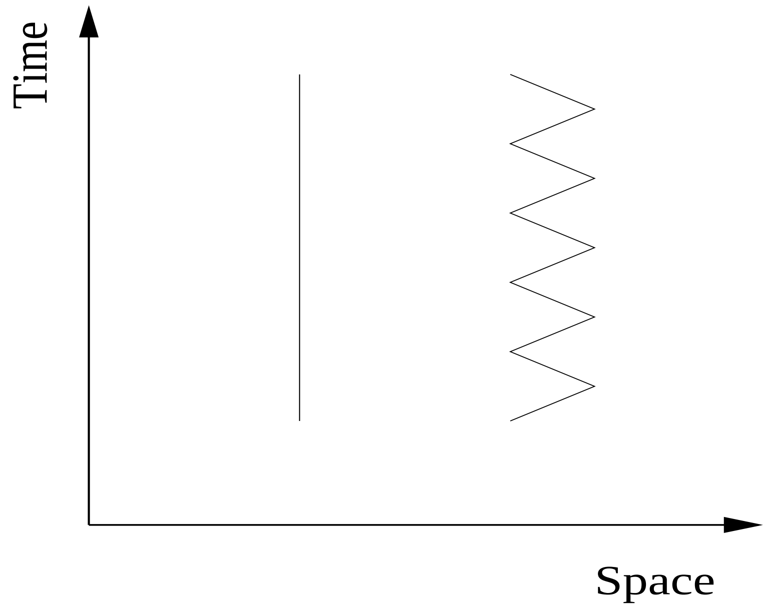

Fig. (4.2). Illustrating the idea of a worldline in a spacetime diagram. See the text for details.

4.3 The Robertson–Walker Metric

In relativity the three dimensions of space and single dimension of time are unified into a four–dimensional manifold known as spacetime. A point in spacetime is then referred to as an *event* and corresponds to something that occurs at a particular time and at a particular place. Motion in the universe can then be represented in terms of a spacetime diagram. One example is shown in Fig. (4.2), where two of the spatial dimensions are suppressed for simplicity, and time runs vertically upwards. A path in spacetime is called a *worldline*. Consider your worldline during a typical cosmology lecture. You remain stationary at your desk during the lecture, but move forwards in time. Your path can therefore be represented by the vertical line. The lecturer, on the other hand, follows (schematically) a zig–zag path as he walks back and forth in front of the blackboard.

In general relativity, the effect of the force of gravity is to *curve* spacetime. The curvature of spacetime then tells matter how to move. The classic analogy is to model the force of gravity in the universe with a sheet of polythene. Consider a hypothetical observer such as an ant who moves about on the surface of such a sheet. The ant can move forwards or backwards and also to its left or right, but cannot move up or down. As far as it is concerned, the universe consists of the two-dimensional surface of the polythene. This surface represents the 'space' of the ant's universe.

Einstein's insight was to realize that spacetime becomes warped when a massive object such as a star is present. Since all massive objects are affected by gravity, this means that the force of gravity can be described in terms of the distortion of spacetime. For example, what would happen if we were to place a heavy ball onto

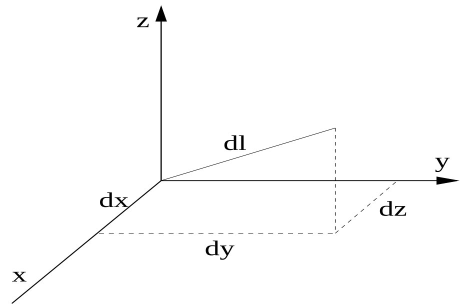

Fig. (4.3). The incremental distance dl is determined by Pythagoras' theorem in ordinary, three–dimensional space.

the sheet? The polythene would become distorted and the space would no longer appear flat. Instead, it will be *curved.* If the ant were to roll a smaller ball onto the polythene in the vicinity of the larger object, it would see the former begin to move towards the latter. This is precisely what happens when the force of gravity operates between large and small objects.

The idea, then, is that we can think of gravity in one of two ways. The force of gravity due to the larger ball acts on the approaching smaller ball. This causes the smaller ball to be attracted towards the larger one. Alternatively, the presence of the large ball causes the polythene to become curved and this alters the path of the smaller ball. Either way the smaller ball moves towards the larger one. The amount of matter in a particular region determines to what extent the space is distorted. A bigger mass results in more distortion. This means that the deflection of small objects from a straight line path is more pronounced and the strength of gravity is stronger, as expected. In short, *the force of gravity is equivalent to a curving of the spacetime.*

Describing gravity in this way is rather like trying to putt a golf ball. If the green were precisely flat, the task would be easy since we would simply have to knock the ball directly towards the hole. However, the bumps and dips typically present on the green imply that the correct path for the golf ball is not necessarily a straight line. Instead the golf ball will follow a curved trajectory as it falls into the dips and moves over or around the bumps. Such effects must be accounted for when aiming the putter. Similarly, we must account for the distortions in spacetime that arise due to massive objects, such as galaxies, in the universe.

The focus for us when considering observational aspects of cosmology is how distances over cosmic scales can be quantified. In ordinary Euclidean space, the distance between two points is given simply by Pythagoras' theorem (Fig. (4.3)):

$$dl^2 = dx^2 + dy^2 + dz^2 \tag{4.7}$$

In special relativity, where the effects of gravity are ignored, one of fundamental postulates is that the speed of light is *constant* regardless of the (constant) velocity of the observer. It can be shown that this requires an incremental distance in spacetime to be given by

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{4.8}$$

This quantity is known as the *metric* of spacetime. The key difference to the Pythagorian result of ordinary Euclidean space, Eq. (4.7), is the presence of the minus sign in front of the dt^2 term. It turns out that this ensures that the speed of light is indeed invariant. In particular, it implies that the quantity

$$ds^2 = 0 \tag{4.9}$$

for light, since it then follows from Eq. (4.8) that $c = dl/dt$, as required.

It can be further shown that in general relativity, the only metric for spacetime that is consistent with the conditions of isotropy and homogeneity – as required by the Cosmological Principle – must have the form

$$ds^2 = -c^2dt^2 + a^2(t)dl^2(x, y, z) \tag{4.10}$$

where the function $a(t)$ depends only on time and dl^2 depends only on the spatial variables. In general relativity, we may view space as a surface of constant time in spacetime with $dt = 0$. It then follows that dl^2 represents the metric of space in the universe. We must therefore determine the possible forms for this metric.

The first point to note is that homogeneity implies that spatial derivatives necessarily vanish. Thus, the curvature of space must be constant throughout the universe (on sufficiently large scales). This implies that it can either be positive, zero, or negative – there are just three possibilities.

Now, the surface of spacetime is four–dimensional and we do not have the ability to visualise such a surface. To proceed, therefore, it is helpful to go to a two–dimensional analogy. The three possibilities are shown in Fig (4.4) (handed out in lectures). Perhaps the simplest such example is the surface of a sphere. This has constant, positive curvature, as follows since any point on the surface is equivalent to any other. A universe with positive curvature is often referred to as a *closed* universe. The case with vanishing curvature just corresponds to flat space and such a universe is known as a *flat* universe. Finally, if the curvature is negative, the shape of the surface corresponds to that of a horse's saddle – such a surface extends to infinity and is often label a *hyperbolic* surface (as opposed to a spherical surface with positive curvature). A universe with negative curvature is known as an *open* universe.

What we need, therefore, is an expression for the metric on these surfaces of constant curvature. We consider the surface of a sphere as an example. Although such a surface is two–dimensional, we can think of it as being embedded in ordinary three–dimensional space (see the top diagram of Fig. (4.4)). As before, we have

$$dl^2 = dx^2 + dy^2 + dz^2 \tag{4.11}$$

but now we are restricted to the surface of the sphere, whose equation is

$$x^2 + y^2 + z^2 = R^2 = \text{constant} \tag{4.12}$$

where R represents the radius of the sphere.

What we are looking for is an expression for the metric on the surface of the sphere expressed in terms of only two coordinates. To this end, it is convenient to work with cylindrical coordinates, (r, θ, z), as defined in terms of the cartesian coordinates (x, y, z) in Fig (4.5) (handed out in the lectures):

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \tag{4.13}$$

It follows that

$$x^2 + y^2 = r^2 \tag{4.14}$$

and that

$$dx^2 + dy^2 = dr^2 + r^2 d\theta^2 \tag{4.15}$$

Substituting Eq. (4.15) into Eq. (4.11) implies that

$$dl^2 = dr^2 + r^2 d\theta^2 + dz^2 \tag{4.16}$$

We now have to eliminate the dependence on the variable z. This is achieved by noting from Eq. (4.12) that the radius is constant, and so, after substituting in Eq. (4.14), we find that

$$r^2 + z^2 = R^2 \quad \Longrightarrow \quad dz^2 = \frac{r^2 dr^2}{z^2} = \frac{r^2 dr^2}{R^2 - r^2} \tag{4.17}$$

Eq. (4.17) may then be substituted into Eq. (4.16):

$$dl^2 = \frac{dr^2}{1 - (r/R)^2} + r^2 d\theta^2 \tag{4.18}$$

This is the metric for the surface of a two–dimensional sphere.

This can be generalized to the three–dimensional case, by working with spherical polar coordinates (r, θ, ϕ), as defined in Fig (4.5). Question 1 of Exercise Sheet 3 takes you through the steps of the derivation – the method is largely identical to that of the two–dimensional case derived above, but the algebra is a little more involved due to the presence of the extra coordinate. The final result is

$$dl^2 = \frac{dr^2}{1 - (r/R)^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{4.19}$$

This represents the space part of the spacetime metric for a closed universe. The spacetime metric itself is therefore deduced by substituting Eq. (4.19) into Eq. (4.10)

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1 - (r/R)^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{4.20}$$

We may set $R = 1$ without loss of generality by an appropriate rescaling. Furthermore, by allowing for spaces with zero and negative curvature, we arrive at the metric for a general isotropic and homogeneous universe. This is known as the Robertson–Walker metric, after the two relativists who discovered it in the early 1930's. The Robertson–Walker metric is

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] \tag{4.21}$$

where $k = +1$ for positive curvature, $k = 0$ for zero curvature and $k = -1$ for negative curvature.

An important point to emphasize is that the constant k that appears in the Robertson–Walker metric (4.21) is the *same* constant k that appears in the curvature term of the Friedmann equation (2.27). Moreover, the function $a(t)$ is the same as the function $a(t)$ arising in the Friedmann equation, i.e., it is the scale factor of the universe[10].

Finally, there is an alternative and convenient way of expressing the Robertson–Walker metric in terms of a new variable χ defined such that

$$r \equiv \begin{cases} \sin\chi & \text{if } k = +1 \\ \chi & \text{if } k = 0 \\ \sinh\chi & \text{if } k = -1 \end{cases} \tag{4.22}$$

This implies that Eq. (4.21) may be written as

$$ds^2 = -c^2 dt^2 + a^2(t) \left[d\chi^2 + S_k^2(\chi) \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] \tag{4.23}$$

where

$$S_k(\chi) \equiv \begin{cases} \sin\chi & \text{if } k = +1 \\ \chi & \text{if } k = 0 \\ \sinh\chi & \text{if } k = -1 \end{cases} \tag{4.24}$$

The variables (r, χ, θ, ϕ) are comoving coordinates, i.e., they do not vary for an observer moving with the Hubble expansion. For $k = 0$, $r = \chi$ plays the role of the radial coordinate and $0 \leq r \leq \infty$. Likewise, for $k = -1$, $0 \leq r \leq \infty$ and $0 \leq \chi \leq \infty$. For $k = +1$, on the other hand, r runs from 0 to 1 and χ runs from 0 to π.

[10]In the relativistic derivation of the Friedmann equation, the starting point is the Robertson–Walker metric (4.21). In general, the Einstein field equations are ten, second–order partial differential equations in the spacetime metric – these reduce to the Friedmann equation when the metric has the Robertson–Walker form.