5 Physical Cosmology II

The focus of this Section is to discuss what we can learn about the universe by measuring the Hubble and Ω parameters, H_0 and Ω_0.

5.1 The Collapse of a Radiation Dominated Universe

In this subsection we investigate what will happen to the universe in the future. If the universe is open or flat ($\Omega_0 \leq 1$), it will expand forever, gradually becoming progressively less dense. Stars will eventually burn out and new star formation will become more and more difficult, until it eventually ceases altogether. The temperature of the cosmic radiation will continue to fall as its wavelength increases with the expansion. Ultimately, the universe reaches a 'heat death'.

If, on the other hand, the universe is closed, $kc^2 = +1$, the two terms on the right–hand side of the Friedmann equation (2.27) will cancel each other for a given value of the scale factor. At this instant, $H = \dot{a}/a = 0$. Since the left–hand side of this equation must remain positive, this implies that the universe then enters up on a phase of contraction and subsequently collapses into a 'big crunch'. Either way the long term prospects for life in the universe are not promising.

It is of interest to estimate when the big crunch will happen in a closed universe. To determine this, we need to solve the Friedmann equation. It is algebraically easier to consider a radiation–dominated universe first[11]. The more physically interesting case of a pressureless universe (corresponding to our universe) can be handled in a similar fashion, but the algebra is slightly more involved.

The Friedmann equation for a spatially closed, radiation dominated–universe is

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\left(\frac{\rho_0 a_0^4}{a^4}\right) - \frac{1}{a^2} \tag{5.1}$$

The maximum size of such a universe is

$$a_{\rm max} = \sqrt{\frac{8\pi G \rho_0 a_0^4}{3}} \tag{5.2}$$

Substituting Eq. (5.2) into (5.1) and rearranging implies that

$$a^2 \left(\frac{da}{dt}\right)^2 = a_{\rm max}^2 - a^2 \tag{5.3}$$

We solve this equation by defining a new time variable in terms of t in such a way that the form of the equation changes to a more familiar form, namely that of a

[11]Although we did not cover this example in the lectures, it is a worthwhile exercise to follow the working as it is similar to that of the pressureless universe considered in the next subsection.

simple harmonic oscillator. The appropriate substitution is to define φ such that

$$dt \equiv a d\varphi \qquad \Longrightarrow \qquad \frac{d}{dt} = \frac{1}{a}\frac{d}{d\varphi} \tag{5.4}$$

Substituting Eq. (5.4) into Eq. (5.3) then implies that

$$\left(\frac{da}{d\varphi}\right)^2 + a^2 = a^2_{\rm max} \tag{5.5}$$

Now, recall from Section 2 that the (Newtonian) derivation of the Friedmann equation followed as a direct result of the conservation of energy, Eq. (2.8). Eq. (5.5) is just the Friedmann equation (5.1) rewritten in terms of new variables. However, it now has the same form as the energy equation for a simple harmonic oscillator, such as a mass oscillating at the end of a spring. If y is the vertical displacement of the spring from its equilibrium position, the restoring force on the mass is proportional to the amount that the spring is stretched, $-\kappa y$, where the constant κ represents the 'stiffness' of the spring. Thus, Newton's second law ($F = ma$) implies that the force on the object of mass, m, is given by

$$m\frac{d^2y}{dt^2} = -\kappa y \tag{5.6}$$

How does this relate to the Friedmann equation (5.5)? Differentiating Eq. (5.5) with respect to φ implies that

$$\frac{d^2a}{d\varphi^2} = -a \tag{5.7}$$

and comparison between Eqs. (5.6) and (5.7) then allows us to draw a direct analogy between a closed, radiation–dominated universe and an oscillating spring. The size of the universe (the scale factor, a) corresponds to the displacement of the mass at the end of the spring and the 'spring' (universe) oscillates as if the mass were equal to the spring stiffness, $m = \kappa$. The maximum displacement of the spring then corresponds to the maximum size attainable by the universe, $a_{\rm max}$, as given in Eq. (5.2).

The solution to Eq. (5.5) can be written in terms of a trigonometric function:

$$a = a_{\rm max} \sin\varphi \tag{5.8}$$

where we have chosen the constant of proportionality so that $a = 0$ when $\varphi = 0$. The physically relevant range of φ is $0 \le \varphi \le \pi$, since this ensures $a \ge 0$. The dependence of time t on the variable φ is then determined by integrating Eq. (5.4):

$$t = \int a \, d\varphi = C - a_{\rm max} \cos\varphi \tag{5.9}$$

where C is the constant of integration. We have freedom in choosing the value of this constant, but as we want the origin of time ($t = 0$) to correspond to the vanishing of the universe ($a = 0$), it is sensible to choose $C = a_{\rm max}$. Hence,

$$t = a_{\rm max}(1 - \cos\varphi) \tag{5.10}$$

Finally, substituting Eq. (5.10) into (5.8) allows us to express the variation of the scale factor with respect to time,

$$a = \left[2a_{\rm max}t - t^2\right]^{1/2} \tag{5.11}$$

We deduce, therefore, that a radiation–dominated universe recollapses after a time

$$\tau = 2a_{\rm max} \tag{5.12}$$

5.2 The Collapse of a Pressureless Universe

Of more interest is what will happen in the future to a universe dominated by pressureless matter, since this appears to represent the matter content of our universe at the present epoch. We follow the method of the radiation–dominated universe described in the previous subsection.

The Friedmann equation is

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\frac{\rho_0 a_0^3}{a^3} - \frac{1}{a^2} \tag{5.13}$$

It is convenient to express the constants of proportionality in this equation in terms of the two observable parameters $\{\Omega_0, H_0\}$. This can be achieved via the Friedmann equation as written in terms of the Ω–parameter, Eq. (4.4). In particular, evaluating quantities at the present epoch implies that

$$a_0 H_0 = \frac{1}{\sqrt{\Omega_0 - 1}} \tag{5.14}$$

Moreover, the definition of the Ω–parameter itself implies that

$$\Omega_0 = \frac{8\pi G\rho_0}{3H_0^2} \tag{5.15}$$

By substituting in Eqs. (5.14) and (5.15), Eq. (5.13) can be written in the form

$$a\left(\frac{da}{dt}\right)^2 = \frac{1}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} - a \tag{5.16}$$

It follows that the point of maximum expansion is reached (at $\dot{a} = 0$) when

$$a_{\rm max} = \frac{1}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} = \frac{\Omega_0}{(\Omega_0 - 1)}a_0 \tag{5.17}$$

Eq. (5.17) tells us that if we live in a closed universe, we can determine when the expansion will be halted (simply) by measuring the present–day values of the Ω– and Hubble parameters. Until a few years ago, a value of $\Omega_0 \approx 2$ was still consistent with the data (it is now much closer to unity), which would imply that the universe would expand to twice its present size before recollapsing. For Ω_0 closer to unity, the universe expands for longer.

At what time will the point of maximum expansion be attained? To determine this we need to solve the Friedmann equation (5.16). We first rewrite this in terms of the φ variable defined in Eq. (5.4):

$$\left(\frac{da}{d\varphi}\right)^2 + a^2 = a a_{\text{max}} \tag{5.18}$$

This would be equivalent to Eq. (5.5) were it not for the additional factor of a on the right hand side. However, we can eliminate this dependence by completing the squares, i.e., by noting that Eq. (5.18) can be written as

$$\left(\frac{da}{d\varphi}\right)^2 + \left(a - \frac{a_{\text{max}}}{2}\right)^2 = \frac{a_{\text{max}}^2}{4} \tag{5.19}$$

Thus, defining a new variable, b, that is related to the scale factor, a, by

$$b \equiv a - \frac{1}{2} a_{\text{max}} \tag{5.20}$$

implies that Eq. (5.19) simplifies to

$$\left(\frac{db}{d\varphi}\right)^2 + b^2 = \frac{1}{4} a_{\text{max}}^2 \tag{5.21}$$

and this is now in the form of a simple harmonic oscillator as desired. (Compare Eq. (5.21) with Eq. (5.5)).

As we want $\varphi = 0$ to correspond to $a = t = 0$, the appropriate solution to Eq. (5.21) is

$$b = -(a_{\text{max}}/2) \cos \varphi \tag{5.22}$$

since substituting this into Eq. (5.20) then implies that

$$a = \frac{a_{\text{max}}}{2}(1 - \cos \varphi) = \frac{1}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} (1 - \cos \varphi) \tag{5.23}$$

The dependence of time, t, on φ is then deduced by integrating Eq. (5.4) as before:

$$t = \int a d\varphi = \frac{1}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} (\varphi - \sin \varphi) \tag{5.24}$$

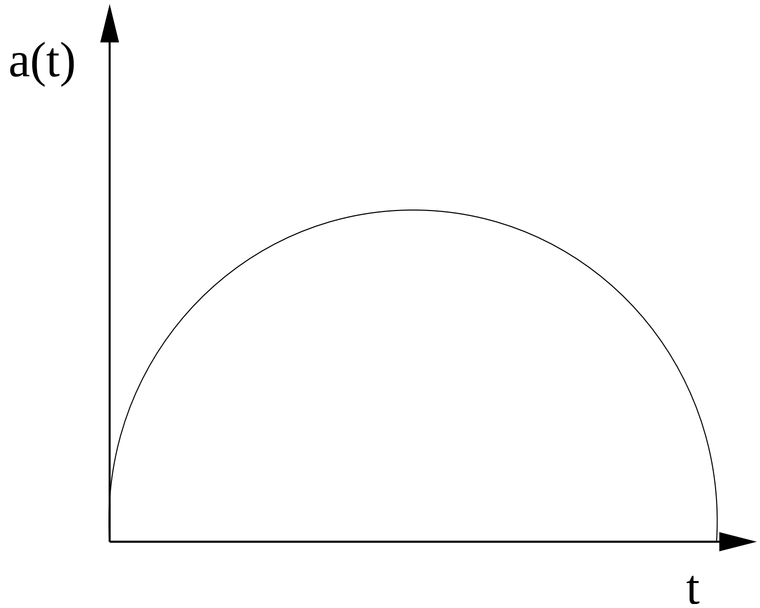

Fig. (5.1). The expansion and subsequent recollapse of a closed, pressureless universe is equivalent to the path mapped out by a point on the circumference of a bicycle wheel that is moving horizontally and without friction. This is known as a cycloid.

where the constant of integration is chosen so that $t = 0$ corresponds to $\varphi = 0$ and $a = 0$.

The variation of the scale factor with φ is shown in Fig. (5.1). As φ varies from 0 to 2π, the scale factor grows, reaches a maximum when $\varphi = \pi$ and then falls back to zero at $\varphi = 2\pi$. The curve is identical to the path mapped out by a point on the circumference of a bicycle wheel as it moves in a horizontal direction without friction. It is known as a *cycloid*. The point of maximum expansion is reached when $\varphi = \pi$. From Eq. (5.24) this corresponds to a time

$$t_{\rm max} = \frac{\pi}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{5.25}$$

Moreover, the time elapsed from the big bang to the big crunch gives the lifetime of this universe. Since the expansion phase is symmetric with the contraction phase, the lifetime, τ, is twice the time it takes to reach its maximum size,

$$\tau = 2t_{\rm max} = \frac{\pi}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{5.26}$$

So, by measuring Ω_0 and H_0, we can predict when the big crunch will happen.

This is not necessarily the end of the story, however. Although the scale factor does indeed vanish when $\varphi = 2\pi$, there is nothing to preventing us mathematically from continuing the solution through to $\varphi > 2\pi$. In this case, once the big crunch is reached, the scale factor grows once more and the universe ‘bounces’ into a new phase

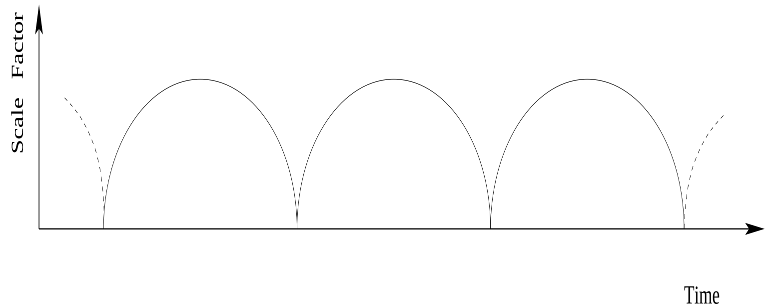

Fig. (5.2). The cycloid solution (5.23) formally corresponds to an infinite series of expanding and contracting phases. A cyclic universe exhibiting such behaviour would have no definite beginning or end.

of expansion. The behaviour shown in Fig. (5.1) is then repeated. In principle this can occur indefinitely, as illustrated in Fig. (5.2). This opens up the possibility that the universe may undergo an indefinite series of cycles interchanging expanding and contracting phases. Such a universe is referred to as a *cyclic universe* and need not necessarily have had a definite beginning (or end). However, from a physical point of view, there are problems in understanding how the transition from big crunch to big bang might be achieved. Indeed, the assumption of zero pressure at arbitrarily small volume is probably not reliable.

5.3 The Age of the Universe

We now show how, for a given matter content, the age of the universe can be determined uniquely in terms of the fundamental observable parameters $\{\Omega_0, H_0\}$.

We expect that the age should depend at least on the quantity H_0^{-1}. Consider a universe whose expansion is linear, $a \propto t$, so that $\ddot{a} = 0$. The time elapsed from the big bang to the present day t_0 would be $T = a_0/\dot{a}_0 = H_0^{-1}$ (see Fig. (5.3)). However, we have seen from the acceleration equation (2.29) that $\ddot{a} < 0$ if $\rho c^2 + 3p > 0$ and so the curve $a(t)$ must be concave, implying that the actual age of the universe must be somewhat less that H_0^{-1}. Hence, H_0^{-1} represents an *upper* limit on the age of the universe. It is often called the *Hubble time*.

A precise expression for the age of the universe follows from the definition of the Hubble parameter, $da/dt = aH$. Viewing this as a separable, first–order differential equation allows us to integrate both sides:

$$t_0 = \int_0^{t_0} dt = \int_0^{a_0} \frac{da}{aH} \tag{5.27}$$

where we explicitly treat the Hubble parameter as a function of the scale factor, $H = H(a)$. Eq. (5.27) is a general expression for the age of the universe. In principle, the age can be determined once we know the form of $H(a)$ and this latter quantity can itself be deduced once we know how the density of matter depends on the scale factor.

Now, in the previous Section we alluded to the fact that the density of (pressureless) matter in the universe is $\Omega_{\mathbf{matter}} \approx 0.3 < 1$. Let us therefore consider an open universe containing just pressureless matter, so that the density is given by Eq. (3.5), $\rho = (\rho_0 a_0^3)/a^3$. The Friedmann equation (2.27) is then

$$H^2 = \frac{8\pi G}{3}\frac{\rho_0 a_0^3}{a^3} + \frac{1}{a^2} \tag{5.28}$$

We may write the constant coefficients in this equation in terms of the present day values Ω_0 and H_0 by recalling from Eq. (4.4) that

$$a_0 H_0 = \frac{1}{\sqrt{1-\Omega_0}} \tag{5.29}$$

From the definition of the Ω–parameter (4.2), we have

$$\Omega_0 = \frac{8\pi G \rho_0}{3H_0^2} \tag{5.30}$$

and substituting Eqs. (5.29) and (5.30) into Eq. (5.28) then implies that

$$a^2 H^2 = \frac{1}{H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}\frac{1}{a} + 1 \tag{5.31}$$

where we have also multiplied both sides by a^2.

It is convenient to define the constant

$$B_0 \equiv \frac{\Omega_0}{H_0(1-\Omega_0)^{3/2}} \tag{5.32}$$

Substituting Eqs. (5.32) and (5.31) into the general expression (5.27) for the age of the universe then yields the equation

$$t_0 = \int_0^{a_0} \frac{da}{[(B_0/a)+1]^{1/2}} \tag{5.33}$$

The problem of determining the age of the universe is now reduced to evaluating the integral[12]. We make the substitution

$$a = B_0 \sinh^2\theta \qquad \Longrightarrow \qquad da = 2B_0 \sinh\theta \cosh\theta d\theta \tag{5.36}$$

[12]We need some identities. They are

$$\cosh^2\theta - \sinh^2\theta = 1 \tag{5.34}$$

$$\int d\theta \sinh^2\theta = \frac{1}{2}(\sinh\theta\cosh\theta - \theta) \tag{5.35}$$

Since $\theta = 0$ when $a = 0$, the integral reduces to

$$t_0 = 2B_0 \int_0^{\theta_0} d\theta \sinh^2 \theta \tag{5.37}$$

Comparison with Eq. (5.35) implies that Eq. (5.37) is in the form of a standard integral and can be evaluated:

$$t_0 = B_0 \left[\sinh\theta \cosh\theta - \theta\right]_0^{\theta_0} \tag{5.38}$$

Finally, noting that

$$\theta_0 = \sinh^{-1}\left(\sqrt{\frac{a_0}{B_0}}\right) = \sinh^{-1}\left(\sqrt{\frac{1-\Omega_0}{\Omega_0}}\right) \tag{5.39}$$

when $a = a_0$, we arrive at the rather complicated expression

$$t_0 = \frac{1}{H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}\left[\frac{(1-\Omega_0)^{1/2}}{\Omega_0} - \sinh^{-1}\left(\sqrt{\frac{1-\Omega_0}{\Omega_0}}\right)\right] \tag{5.40}$$

for the age of the universe.

Note the factor of H_0^{-1} has appeared as we anticipated. So, if we can measure H_0 and Ω_0, we can deduce the age of the universe[13].

Eq. (5.40) is not particularly illuminating. It can be simplified when Ω_0 is not too different from unity, as indicated by observations. Define a parameter ϵ such that

$$\Omega_0 \equiv 1 - \epsilon, \qquad \epsilon \ll 1 \tag{5.41}$$

Then

$$t_0 = \frac{1}{H_0}\frac{1-\epsilon}{\epsilon^{3/2}}\left[\frac{\epsilon^{1/2}}{1-\epsilon} - \sinh^{-1}\left(\sqrt{\frac{\epsilon}{1-\epsilon}}\right)\right] \tag{5.42}$$

By expanding to first–order in ϵ and employing the expansion[14] (5.43), it follows that (see Question 3 of Problem Sheet 3)

$$t_0 = \frac{2}{3H_0}\left[1 + \frac{1}{5}(1-\Omega_0)\right] \tag{5.45}$$

[13]Of course, this assumes that the universe has been dominated by pressureless matter for most of its lifetime. As we see later, the universe was effectively radiation dominated for the first 10,000 years of its life, but this is a small correction so we do not need to concern ourselves with it here. Indeed, we have already touched on this in Section 3.

[14]The series expansion for the inverse hyberbolic sine function for small x is given by

$$\sinh^{-1} x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots \tag{5.43}$$

The expansion of a polynomial around small x ($x \ll 1$) is

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2}x^2 + \ldots \tag{5.44}$$

We can rewrite Eq. (5.45) in terms of years by substituting Eq. (1.5) for the Hubble parameter:

$$t_0 \approx \frac{6.5}{h}\left[1+\frac{1}{5}(1-\Omega_0)\right] \text{ Gyr} \tag{5.46}$$

It is important to note that in the limit $\Omega \to 1$, the age is given by

$$t_0 = \frac{2}{3}H_0^{-1} = 6.5h^{-1} \text{ Gyr} \tag{5.47}$$

In the opposite limit, where the universe is almost empty, $\Omega_0 \to 0$, the first term on the right hand side of Eq. (5.40) dominates, and the age is $t_0 = H_0^{-1}$.

A similar analysis can be performed for closed ($\Omega_0 > 1$) universes. We quote the full expression:

$$t_0 = \frac{1}{H_0}\frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\left[\cos^{-1}\left(2\Omega_0^{-1}-1\right) - \frac{2}{\Omega_0}\left(\Omega_0-1\right)^{1/2}\right] \tag{5.48}$$

It turns out that for Ω_0 close to unity, Eq. (5.48) reduces once more to Eq. (5.45).

The full dependence of the age of the universe on Ω_0 is shown in Fig. (5.4) in units of H_0^{-1}. We see that for a given value of H_0^{-1}, the age of the universe is a *decreasing* function of Ω_0. A large Ω_0 implies more matter is present and hence the expansion rate slows down more rapidly because there is more gravitational attraction between the galaxies. Thus, the universe must have spent less time in decelerating to its present expansion rate (as quantified by the value of H_0^{-1}) than in a universe where Ω_0 is lower.

Now, an important constraint on the age of the universe is that it must exceed the age of the oldest objects within it, since presumably such objects could not have existed before the universe began. Some of the oldest objects in the universe are globular clusters (see Figs. (5.5)–(5.9) handed out in the lectures). These are gravitationally bound objects containing as many as 10^6 stars and are generally located in the outer regions of galactic halos. Stars in a globular cluster have similar chemical compositions implying that they have similar ages. Moreover, the fact that globular clusters are distributed in a spherical halo around the galactic centre indicates that they formed during the early phase of the Galaxy's history at least before the galactic material had formed into a disc. Further evidence that globular clusters are old is deduced from their chemical composition. The abundances of heavier elements in globular clusters is significantly lower than in second generation stars such as the Sun, implying that they formed out of mainly primordial matter (rather than matter from supernovae remnants).

The age of a globular cluster is deduced from the Hertzsrung-Russel (HR) diagram that plots a star's luminosity with its temperature (cf. your stellar structure lectures). Both these quantities are determined by the star's mass. Hence the position of a star on the HR diagram is determined by its mass (see Fig (5.7)). Most stars lie on the Main Sequence – a diagonal band running from the bottom–right to the top–left of the diagram. This is the phase where hydrogen in the stellar core is fused to helium. More massive stars are more luminous and evolve off the main sequence more rapidly than lower mass stars.

Now, stars in a given globular cluster have a similar history, in that they have a similar chemical composition. They therefore differ only in their masses. Consequently, the stars in a newly formed globular cluster are distributed along the main sequence over a range of masses. As time proceeds, the more massive stars turn off the main sequence sooner than their lower mass siblings. Thus, the older the globular cluster, the higher the number of stars that have turned off the main sequence. The turn–off point (TO) is then lower in the diagram, i.e., the main sequence band is *shorter*. The age of a globular cluster is then determined by the position of the turn–off point (Fig. (5.8)). A numerical estimate for the age is deduced by fitting a theoretical model for the evolution of the stars with the observed data.

It is found that typically globular cluster ages are in the range

$$t_{\rm globclust} = 12 \pm 1\,{\rm Gyr} \tag{5.49}$$

How does this compare with the inferred age of the universe as derived above? Taking the central value of 12 Gyr consistent with observation, we deduce from Eq. (5.47) that a flat ($k = 0$), pressureless universe has an age in excess of this only for $h < 0.54$. However, this is barely consistent with the range of values for the Hubble parameter discussed in Section 1. Indeed, the favoured value deduced from the Hubble Space Telescope ($h = 0.72$), implies $t_0 \approx 9\,{\rm Gyr} < t_{\rm globclust}$. Moreover, taking the upper value of 13 Gyr for globular cluster ages implies that $h < 0.5$ for $t_0 > t_{\rm globclust}$ and this is certainly *not* observed.

Does this indicate we that are not living in a flat universe with $\Omega_0 = 1$? Possibly – the favoured value for the Ω–parameter inferred from galaxy cluster dynamics is $\Omega_{\rm matter} = 0.3$. Putting this value into the full expression (5.40) implies that $t_0 \approx 7.9h^{-1}$ Gyr and, for a favoured value of $h = 0.65$, this yields an age of 12 Gyr that is consistent with globular cluster ages.

However, this is by no means the end of the story. Of course, it is possible that globular cluster ages have been miscalculated. On the other hand, from the cosmological perspective, the above analysis rested on the assumption that the matter in the universe is pressureless. As we shall see in later Sections, there are strong theoretical reasons for supposing that we do indeed live in a universe where $\Omega_0 = 1$ (see the Section on inflationary cosmology). We can conclude at this stage that for the measured values of the Hubble constant, the age of a flat, pressureless universe is probably too low to be consistent with globular cluster ages. If the universe really is flat, then it is likely that there are other, more exotic types of matter present in the universe that we have yet to consider. It turns out that this is probably the case, as we discuss in the next Section.

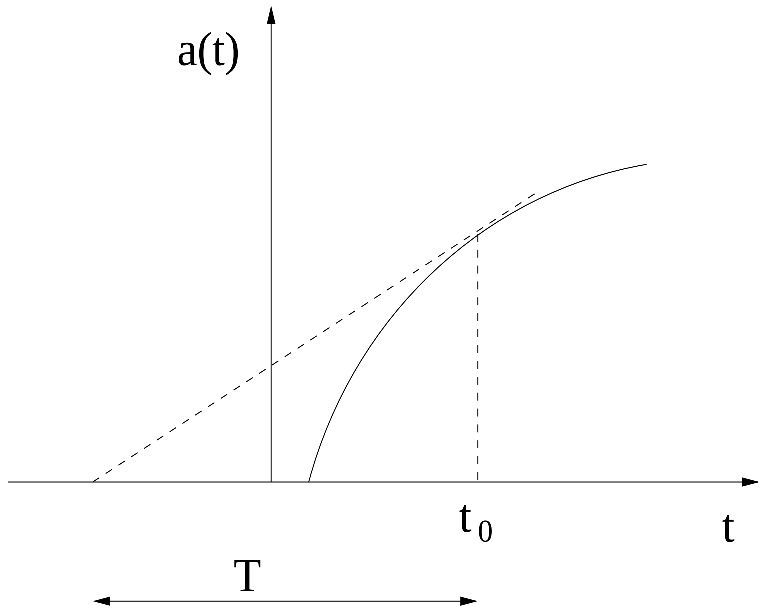

Fig. (5.3). The Hubble time H_0^{-1} represents an upper limit on the age of the universe.

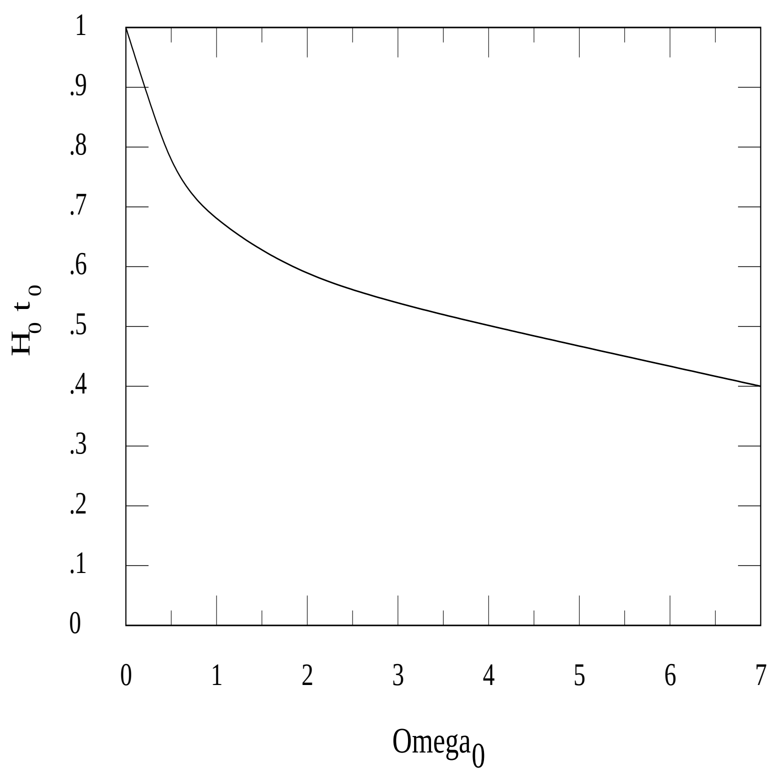

Fig. (5.4). The age of the universe containing pressureless matter as a function of Ω_0 measured in units of H_0^{-1}. The age decreases with increasing Ω_0. Since the era of mater–radiation equality occurs relatively soon after the big bang, this measures the age of the universe.