6 Distance Scales and the Cosmological Constant

6.1 Distance in the Universe and the Deceleration Parameter

The Robertson–Walker metric (4.23) is the tool we need to determine distances in the universe. We first need to relate the comoving coordinate to a parameter that is in principle observable. An obvious choice is to relate it to redshift, since the redshift of an object increases with its distance from us. The redshift is an important observable because it is easily measurable (directly from the spectral lines) and also provides a model independent measure of distance.

Firstly, we note that in an isotropic universe, a light pulse must follow a radial path such that the angular variables in the metric, θ and ϕ, are constant, $d\theta = d\phi = 0$. Moreover, recall from Section 4.2 that the condition in special relativity for the speed of light to be invariant is that $ds^2 = 0$ [cf. Eq. (4.9)]. From the Robertson–Walker metric (4.23) we therefore have

$$ds^2 = -c^2dt^2 + a^2(t)d\chi^2 = 0 \tag{6.1}$$

for a light wave, or in other words,

$$ad\chi = -cdt = -c\frac{dt}{da}da = -c\frac{da}{\dot{a}} = -c\frac{da}{aH} \tag{6.2}$$

where the minus sign is chosen because we are interested in light moving towards us from a distant object.

Eq. (6.2) is related to redshift via the definition (3.37):

$$1 + z = \frac{a_0}{a(t)} \qquad \Longrightarrow \qquad dz = -\frac{a_0}{a^2}da \tag{6.3}$$

which implies that

$$ad\chi = \frac{c}{H}\frac{a}{a_0}dz \tag{6.4}$$

and hence that

$$a_0d\chi = c\frac{dz}{H(z)} \tag{6.5}$$

where it is convenient to view the Hubble parameter as a function of redshift. Eq. (6.5) is an extremely important equation – it relates the comoving radial coordinate, χ, directly to redshift, z.

The next task is to relate the comoving coordinate to a measure of distance, since Eq. (6.5) will then allow us to relate this to redshift. However, there is an ambiguity with measuring distance in the universe, since the universe is expanding and the speed of light is finite. To illustrate this, consider a distant galaxy. Suppose the light we detect from this galaxy at the present time t_0 was emitted at a time t_e. How can we agree on the distance separating this galaxy from ours? Do we measure the distance when the light was emitted by the galaxy or when it was received by us? One approach is to define what is called the[15] *proper distance*, denoted by D_P. *The proper distance is the length that one would measure between two points if the measurement could be made instantaneously.* That is, the proper distance is determined from the Robertson–Walker metric under the condition that $dt = 0$. Thus, for a radial measurement we have from the metric (4.23) that $dD_P \equiv ds = a(t)d\chi$ and, since we are interested in measurements made at the present time, t_0, the proper distance is related to the comoving coordinate by

$$D_P = a_0\chi \tag{6.6}$$

where, by convention, we assume that we are located at the origin of the comoving coordinate system $\chi = 0$.

We have actually met Eq. (6.6) previously in Section 2, as Eq. (2.7). Indeed, it is helpful to go back to our analogy of the expansion of the universe in terms of the inflation of a balloon, where the expansion of space is interpreted as the stretching of the balloon's elastic surface. At a specific moment in time, take a ruler and measure the distance along the surface of the balloon between two points on the elastic. The distance you measure then corresponds to the proper distance.

The desired relationship between proper distance and redshift is now deduced by integrating Eq. (6.5) and substituting into Eq. (6.6). Since $z = 0$ at $t = 0$ and $z = z$ at $t = t_e$ (the time when the light is emitted by the distant object), it follows that

$$D_P = c\int_0^z \frac{dz}{H(z)} \tag{6.7}$$

where we implicitly view the Hubble parameter as a function of redshift, $H = H(z)$. (This is equivalent to what we have been doing up to now, where we have viewed the Hubble parameter as a function of the scale factor, since redshift and scale factor are directly related by Eq. (6.3)).

In principle, if we knew the type of matter present in the universe – that is, the equation of state, Eq. (2.17) – we could solve the conservation equation(s) (2.28) to determine how the densities of the different types of matter present in the universe varied with the scale factor, and by implication, redshift. This would then allow us to determine how the Hubble parameter varies with redshift. If the integral in Eq. (6.7) can then be evaluated, this will allow us to determine how the proper distance of a light source varies with redshift.

However, we are interested in keeping the analysis as general as possible at this stage without specifying the form of the matter. An approximate expression, valid at small redshifts, can be derived by expanding the function $H = H(z)$ as a Taylor series around the present epoch, corresponding to $z \ll 1$, i.e., to $a \approx a_0$. In general,

[15]The terminology 'proper distance' is a technical definition relevant to issues arising in general relativity. These need not concern us here.

the Taylor expansion of a continuous function $y(x)$ about the point x_0 is

$$y(x) = y(x_0) + \left(\frac{dy}{dx}\right)_{x=x_0}(x - x_0) + \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)_{x=x_0}(x - x_0)^2 + \ldots \qquad (6.8)$$

where $(\ldots)_{x=x_0}$ implies the bracketed quantity is to be evaluated at $x = x_0$.

To zeroth–order, we have simply that $H(z) \approx H_0 =$ constant. Substituting this into Eq. (6.7) and evaluating the integral yields

$$D_P = \frac{c}{H_0}z \qquad (6.9)$$

This is just Hubble's law, Eq. (1.3)! Thus, we have effectively derived Hubble's law directly from the Robertson–Walker metric. This serves to illustrate two important points. Firstly, that Hubble's law applies to proper distances and secondly, that it is really only an approximation that holds for galaxies that are sufficiently close to ours, i.e., for galaxies that have sufficiently small redshift.

We may derive a more accurate dependence of proper distance on redshift by expanding $H = H(z)$ to first–order:

$$H(z) = H_0 + \left(\frac{dH}{dz}\right)_0 z + \mathcal{O}(z^2) \qquad (6.10)$$

Invoking Eq. (5.44) implies that

$$H^{-1}(z) \approx H_0^{-1}\left[1 - \frac{1}{H_0}\left(\frac{dH}{dz}\right)_0 z\right] \qquad (6.11)$$

However,

$$\frac{dH}{dz} = \frac{dt}{dz}\frac{dH}{dt} = -\left(\frac{a}{a_0H}\right)\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) \qquad (6.12)$$

It proves convenient at this stage to introduce a new parameter, q, defined by

$$q \equiv -\frac{a\ddot{a}}{\dot{a}^2} = -\frac{\ddot{a}}{aH^2} \qquad (6.13)$$

This is the *deceleration parameter* we advertised at the outset of Section 4 as the third observable parameter of primary importance in cosmology. It derives its name due to its dependence on the second time derivative of the scale factor.

Thus, substituting Eqs. (6.12) and (6.13) into Eq. (6.11) implies that

$$H^{-1}(z) \approx H_0^{-1}\left[1 - (1 + q_0)z\right] \qquad (6.14)$$

where $q_0 = q(t_0)$ and substituting this expression into Eq. (6.7) and evaluating the integral then yields

$$D_P \approx \frac{c}{H_0}\left[z - \frac{1}{2}(1 + q_0)z^2\right] \qquad (6.15)$$

We see that there is a quadratic correction to Hubble's law that becomes important at higher redshifts. If we can measure the extent of this deviation, this allows us to determine the numerical value of q_0.

The third step is to relate proper distance to an observable quantity. In practice, if we know the luminosity or intrinsic brightness, L, of an object, we can deduce its distance from us by measuring the flux, f, that we receive. (The flux is the amount of energy received per unit area per unit time). Consider first the case where space is not expanding and the geometry of space is Euclidean (i.e. not curved). In this case, the observed flux is given by

$$f = \frac{L}{4\pi d^2} \tag{6.16}$$

if the object is a distance d away.

However, in reality space is expanding and may also be curved. How do we account for these effects? If space is curved, the way light spreads out from the source is altered. In a closed universe, for example, the light spreads out over a smaller area. To see this, consider Fig. (6.1). The radius of the equatorial circle (line of latitude) with constant radial coordinate r is given by $R \sin\chi$, where R is the radius of the sphere. Hence, the area of the sphere covered by the light pulse at a coordinate distance χ is $4\pi a^2 \sin^2\chi = 4\pi a^2(t) S_k^2$, where S_k is defined in Eq. (4.24). Similar arguments apply for an open universe, in that the correction factor is once more determined by S_k.

However, the expansion of the universe must also be accounted for. This introduces two further corrections. Firstly, as we saw in Section 3.3, the wavelength of radiation is stretched due to the expansion of the universe in direct proportion to the scale factor. By the time the radiation reaches us, therefore, its energy has been reduced by a factor $(1+z)$. Secondly, the arrival rate of the radiation (as measured by the frequency of its peaks) is also reduced by the factor $(1+z)$. So, overall, the actual flux of an object received by us is given by

$$f = \frac{L}{4\pi [a_0 S_k (1+z)]^2} \tag{6.17}$$

By analogy with the static expression (6.16), it is natural to define what is termed the *luminosity distance*, D_L:

$$f \equiv \frac{L}{4\pi D_L^2} \quad \Longrightarrow \quad D_L \equiv a_0 S_k(\chi)(1+z) = \left(\frac{L}{4\pi f}\right)^{1/2} \tag{6.18}$$

In principle, if we know the luminosity of the source, we can immediately determine the corresponding luminosity distance by measuring the observed flux.

For an object at sufficiently small redshift, its comoving coordinate satisfies $\chi \ll 1$ and we may approximate $S_k \approx \chi$ in the two cases where $kc^2 = \pm 1$. In this limit,

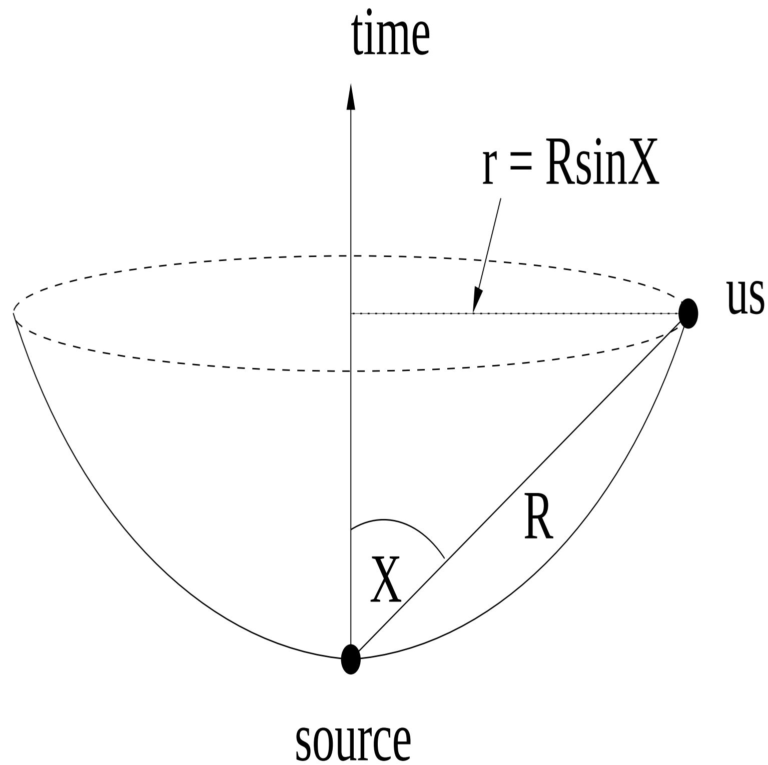

Fig. (6.1). The radius of the the sphere covered by the light pulse in a closed universe is reduced by the factor of $\sin\chi$ to $r = R\sin\chi$, where R is the radius of the sphere.

comparison with Eq. (6.6) implies that we may relate the proper distance to the luminosity distance:

$$D_L \approx D_P(1+z) \tag{6.19}$$

This level of approximation is consistent with the Taylor expansion we performed in deriving Eq. (6.15). Substituting Eq. (6.19) into Eq. (6.15) then leads us to our desired result:

$$D_L \approx \frac{c}{H_0}\left[z + \frac{1}{2}(1-q_0)z^2\right] \tag{6.20}$$

This equation relates the luminosity distance to redshift. The deviations from the linear relation are uniquely determined by the deceleration parameter, q_0.

Given the type of matter present in the universe (that is, the equation of state (2.17)), the value of Ω_0 can be related to the value of the deceleration parameter, q_0, defined in Eq. (6.13). Since q_0 is defined in terms of the second time derivative of the scale factor, we may substitute the acceleration equation (2.29) into Eq. (6.13):

$$q = \frac{4\pi G}{3H^2}\left(\rho + \frac{3p}{c^2}\right) \tag{6.21}$$

Substituting for the critical density (4.1) and assuming the universe is dominated by pressureless matter, $p = 0$, then implies that

$$q = \frac{1}{2}\Omega \tag{6.22}$$

and so at the present epoch, $q_0 = \Omega_0/2$.

Thus, by determining the value of the deceleration parameter through a deviation from Hubble's law at higher redshift, as in Eq. (6.20), we may in principle determine the value of Ω_0. However, care should be taken here because the conclusions are strongly dependent on the type of matter present in the universe. For example, it can be shown that for a radiation–dominated universe, $q_0 = \Omega_0$ (see problem sheet IV). Consequently, we must be sure that we understand what matter is present in the universe before employing this method.

Our expressions for luminosity distance have been approximate, in the sense that that they are only valid for sufficiently small redshifts. On the other hand, we have made no assumption yet about the type of matter present in the universe. It turns out that an exact expression can be found for the case of a pressureless universe with arbitrary value of Ω_0 (see Appendix D for the derivation). The result is

$$D_L = \frac{2c}{H_0}\frac{\Omega_0 z + (\Omega_0 - 2)\left[\sqrt{1+\Omega_0 z} - 1\right]}{\Omega_0^2(1+z)} \tag{6.23}$$

The full dependence of luminosity distance with redshift is shown in Fig. (6.2) for $q_0 = \{0, \frac{1}{2}, 1\}$.

6.2 Supernovae as Standard Candles

Perhaps the simplest method of determining the luminosity distance is to identify a class of objects whose intrinsic luminosity is well understood. Such objects are known as 'standard candles'. In recent years a type of supernova explosion – known as type Ia – has been employed as a standard candle and has played a pivotal role in comparing observations with what we expect from our theoretical models.

A type Ia supernovae arises in a binary system, where one of the stars is a white dwarf whose mass has not exceeded the Chandrasekhar limit. However, material from the companion star in the system is accreted onto the white dwarf, thereby increasing the latter's mass above the Chandrasekhar limit. The electron degeneracy pressure is then unable to halt further collapse of the white dwarf and the subsequent contraction of the core results in the fusion of carbon and oxygen into stable nuclei, primarily iron, nickel and cobalt. This fusion results in a huge release of energy, typically 10^{44} J, in the form of γ–rays. Since all white dwarves are similar in nature, type Ia supernovae have similar light curves and attain a similar level of maximal brightness. Although there are a number of issues still to be resolved, it is believed that the nuclear physics is sufficiently well understood that the luminosity of these objects can be calculated to a reliable accuracy. Consequently, their observed brightness acts as a very good distance indicator – *those that appear dimmer are further away.*

The measured luminosity distance compared with redshift for type Ia supernovae is shown in Fig. (6.3). The top diagram is for objects that are relatively close, out to a redshift $z < 0.1$ and verifies that the large–scale Hubble flow is linear out to these distances, with a best–fit value for the Hubble constant of $H_0 = 64 \pm 3\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, somewhat lower, but still consistent with, the value deduced from the Hubble Space Telescope as discussed in Section 1. The second diagram includes data out to a redshift of $z \approx 1$ and plots the inverse of luminosity distance divided through by redshift. In this parametrization, the linear law corresponds to a horizontal line. The three solid lines correspond to the theoretical curves expected from Eq. (6.23) for a pressureless, open universe with different values of Ω_0. Deviations from the linear law are seen, as expected. What is surprising is that the data disfavours a pressureless universe – certainly the flat model does not fit the data well. The supernovae appear fainter, for a given redshift, than expected in these models. This implies that they are further away. (We will consider the two dashed curves after discussing the cosmological constant).

This is the second example that we have seen that seems to indicate that a flat, pressureless universe may not fit the data (the age problem discussed towards the end of Section 5 was the first example).

If the supernovae are further away that expected, this implies that the universe has been expanding more rapidly than anticipated based on our theoretical assumptions. This suggests that there may be a more exotic type of matter present in the universe that is forcing the galaxies apart, or in other words, is acting as a kind of 'anti–gravity' effect. A cosmological constant can result in such an effect and the possible role that such a term may play in the universe is the topic of the following subsections.

6.3 The Cosmological Constant

We have seen that one of the principle consequences of the Friedmann equation (2.27) is that it implies that the universe can not be static, i.e., it has to be either expanding or contracting, since in general $H \neq 0$. This is to be expected since matter attracts gravitationally. However, at the time that Einstein formulated his theory of General Relativity, the commonly held view was that the universe was indeed static. (This was before Hubble had announced his results in 1929). In an effort to render his theory consistent with the notion of a static universe, Einstein modified his theory by introducing a constant term into the right hand side of the Friedmann equation. This term is referred to as the *cosmological constant* and is usually denoted Λ or ρ_Λ. The Friedmann equation for a universe containing matter and a non–zero cosmological constant is

$$H^2 = \frac{8\pi G}{3}\rho + \frac{8\pi G}{3}\Lambda - \frac{kc^2}{a^2} \tag{6.24}$$

Einstein's idea was to balance the positive curvature of the universe with this constant to ensure that $H = 0$. (See Problem Sheet 2). He abandoned this approach when it became clear that the universe is expanding. However, the idea of a cosmological constant has resurfaced in recent years for somewhat different reasons.

There are different ways of interpreting the cosmological constant from a physical point of view. Mathematically, it arises as a constant of integration when the acceleration equation (2.29) is integrated to yield the Friedmann equation (2.27). In this sense, we may simply regard as a fundamental constant of nature, in the same way as the speed of light in vacuum is fundamental. On the other hand, a cosmological constant may also be interpreted in terms of the energy associated with the vacuum – the vacuum contains no matter, so $\rho = 0$, but the right-hand side of the Friedmann equation (6.24) is still non–zero even when $k = 0$. Indeed, as we have defined Λ in Eq. (6.24), comparison with the first term on the right hand side suggests that we may interpret it in terms of a constant density, $\rho_\Lambda \equiv \Lambda$. Consistency with the conservation equation (2.28) then implies that the effective equation of state is

$$p_\Lambda = -\rho_\Lambda c^2 \tag{6.25}$$

implying that effective pressure is negative for a positive cosmological constant.

When considering a universe containing ordinary matter and a cosmological constant, it is often convenient to introduce separate Ω–parameters for the two components, $\Omega_M = \rho_M/\rho_c$ and $\Omega_\Lambda = \Lambda/\rho_c$, respectively. Then the Friedmann equation (6.24) becomes

$$\Omega_M + \Omega_\Lambda - \frac{kc^2}{a^2H^2} = 1 \tag{6.26}$$

In the limit where the cosmological constant dominates the matter, $\Omega_\Lambda \gg \Omega_M$, the solution to the Friedmann equation (6.24) for $k = 0$ is

$$\frac{\dot{a}}{a} = H = \sqrt{\frac{8\pi G\Lambda}{3}} \qquad \Longrightarrow \qquad a(t) = a_0 \exp\left[H(t - t_0)\right] \tag{6.27}$$

and therefore the expansion is *exponential*. We return to behaviour of this type in the Sections on inflationary cosmology.

6.4 Observational Evidence for a Cosmological Constant

One of the primary implications of a cosmological constant is that is changes the rate of expansion of the universe. It is to be expected, therefore, that it should affect the present–day value of the deceleration parameter (6.13). Indeed, it follows from the acceleration equation (2.29) that when the matter is pressureless, the deceleration parameter can be written as (see Problem Sheet IV)

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \frac{\Omega_M}{2} - \Omega_\Lambda \tag{6.28}$$

In a universe containing just a cosmological constant ($\Omega_M = 0$), this implies that $q < 0$, or equivalently, that the expansion accelerates, $\ddot{a} > 0$, as time progresses.

If the cosmological constant is significant today, one consequence is that it should lead to an observable – and therefore testable – deviation from Hubble's law in Eq. (6.20), primarily because it alters the value of the deceleration parameter, q_0 through Eq. (6.28). In particular, we may rewrite the expression for luminosity distance, Eq. (6.15), in terms of $\Omega_{\Lambda 0}$ such that

$$D_L = \frac{c}{H_0}\left[z + \frac{1}{4}\left(2 - \Omega_{M0} + 2\Omega_{\Lambda 0}\right)z^2\right] \tag{6.29}$$

Over recent years, evidence has mounted in favour of a non–zero cosmological constant from the type Ia supernova redshift surveys. Returning to Fig. (6.3), the two dashed lines correspond to theoretical models for flat universes containing a cosmological constant, where $\Omega_{M0} = 0.3$ and $\Omega_{M0} = 0$ ($\Omega_{\Lambda 0} = 1$), respectively. These fit the data much more closely. The corresponding deviations from Hubble's law are also shown in Fig. (6.4). This figure illustrates the dependence of the effective magnitude of the supernovae with redshift. The magnitude, m_B, can be directly related to luminosity distance and conventionally a higher value for m_B corresponds to a dimmer object, and therefore one that is further away. Supernovae out to a redshift of unity have been observed, corresponding to when the universe was half its present size.

Surprisingly it is found that the supernovae appear dimmer than expected on the basis of a standard cosmology where all matter in the universe is pressureless.

The expected magnitude of the supernovae for different cosmological models have also been included in Fig. (6.4). They are parametrized in terms of the two parameters ($\Omega_{M0}, \Omega_{\Lambda 0}$). The solid lines represent models with no cosmological constant, whereas those with a cosmological constant are represented by the dashed lines. Remarkably, the observations indicate that the supernovae are about 30 % fainter than they are expected to be if the universe contained just pressureless matter and no cosmological constant, i.e., if $q_0 = \Omega_{M0}/2$. Indeed, a universe containing no cosmological constant is strongly disfavoured. (If there is no cosmological constant, the data would have to lie within the region bounded by the solid curves – this does not appear to be the case). The best–fit to the data has $\Omega_{M0} = 0.28$ and $\Omega_{\Lambda 0} = 0.72$ if we assume $\Omega_0 = 1$.

The primary implication of this is that most of the effective density (or equivalently energy) in the universe is due to this cosmological constant. It is often referred to as *dark energy.*

6.5 Age of the Universe with a Cosmological Constant

There is a further benefit of having a non–zero cosmological constant in the universe and this is related to the problem that the age of a flat universe appears to be less than that of typical globular clusters, as we discussed in Section 6. Whereas the gravitational attraction between galaxies tends to hinder the expansion of the universe, a cosmological constant has the opposite effect and acts to increase the rate of expansion. This follows directly from the Friedmann equation (6.24) since for a given density of matter the Hubble parameter is larger than it would be for a vanishing cosmological constant. As a result, it takes longer for the Hubble parameter to decrease to its present observed value and a cosmological constant therefore acts to *increase* the age of the universe for a given observed value of H_0.

Let us quantify this for a flat universe containing pressureless matter and a cosmological constant. The Friedmann equation (6.24) may be written in the form

$$H^2 = \frac{\Omega_{M0} H_0^2 a_0^3}{a^3} + \Omega_{\Lambda 0} H_0^2 \tag{6.30}$$

and substituting Eq. (6.30) into Eq. (5.27) implies that the age of such a universe is given by the integral

$$t_0 = \frac{1}{H_0} \int_0^{a_0} da \frac{a^{1/2}}{\left[\Omega_{M0} a_0^3 + \Omega_{\Lambda 0} a^3\right]^{1/2}} \tag{6.31}$$

To evaluate this integral we make the change of variable to

$$y = \left(\frac{\Omega_{\Lambda 0}}{\Omega_{M0} a_0^3}\right)^{1/2} a^{3/2} \qquad \Longrightarrow \qquad dy = \frac{3}{2}\left(\frac{\Omega_{\Lambda 0}}{\Omega_{M0} a_0^3}\right)^{1/2} a^{1/2} da \tag{6.32}$$

implying that

$$t_0 = \frac{2}{3H_0} \frac{1}{\sqrt{\Omega_{\Lambda 0}}} \int_0^{y_0} \frac{dy}{\left[1 + y^2\right]^{1/2}} \tag{6.33}$$

This is in the form of the standard integral[16] (6.34) and we therefore deduce that

$$t_0 = \frac{2}{3H_0} \frac{1}{\Omega_{\Lambda 0}^{1/2}} \ln\left[\left(\frac{\Omega_{\Lambda 0}}{\Omega_{M0}}\right)^{1/2} + \left(1 + \frac{\Omega_{\Lambda 0}}{\Omega_{M0}}\right)^{1/2}\right] \tag{6.35}$$

Recalling that Eq. (6.26) reduces to $\Omega_{\Lambda 0} + \Omega_{M0} = 1$ for a flat universe allows us to simplify Eq. (6.35) to

$$t_0 = \frac{2}{3H_0} \frac{1}{\sqrt{\Omega_{\Lambda 0}}} \ln\left[\frac{1 + \sqrt{\Omega_{\Lambda 0}}}{\sqrt{1 - \Omega_{\Lambda 0}}}\right] \tag{6.36}$$

and from Eq. (1.5), this is given in units of Gyr by

$$t_0 = \frac{6.5}{h} \frac{1}{\sqrt{\Omega_{\Lambda 0}}} \ln\left[\frac{1 + \sqrt{\Omega_{\Lambda 0}}}{\sqrt{1 - \Omega_{\Lambda 0}}}\right] \text{ Gyr} \tag{6.37}$$

[16] The integral is

$$\int \frac{dx}{\left[1 + x^2\right]^{1/2}} = \sinh^{-1} x = \ln\left[x + \sqrt{1 + x^2}\right] \tag{6.34}$$

and follows by making the substitution $x = \sinh z$.

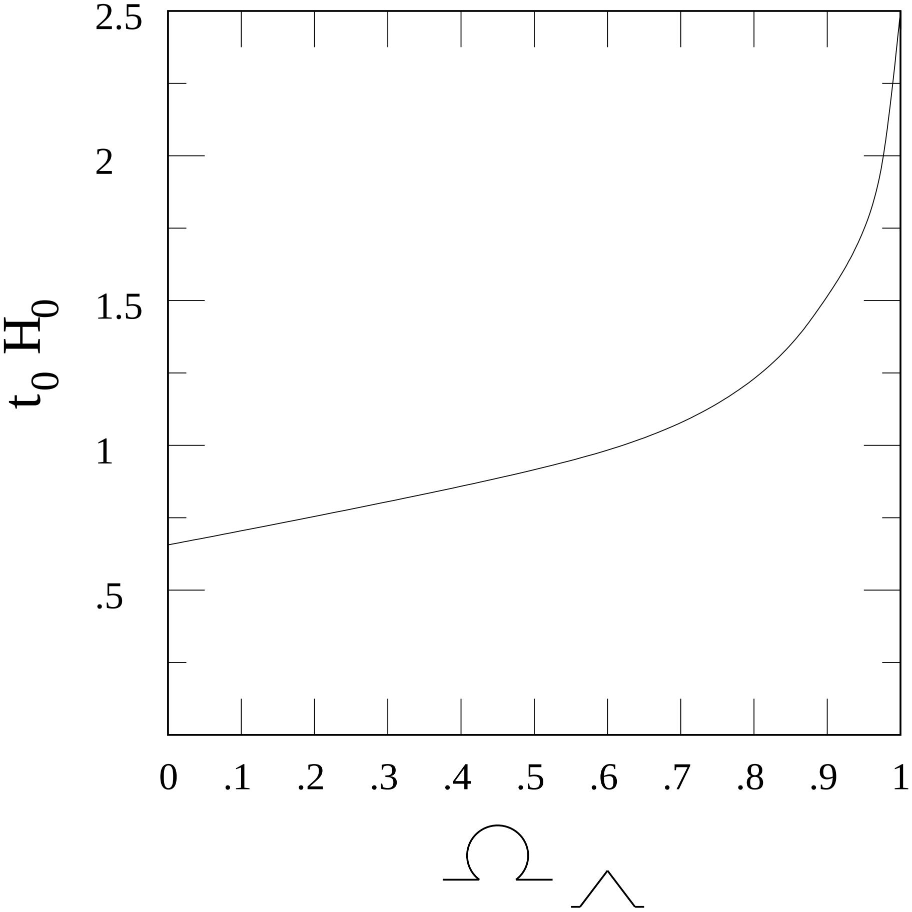

Fig. (6.5). The age of a flat universe with a cosmological constant. For a given value of the Hubble constant, the age increases with increasing Ω_Λ because a cosmological constant increases the expansion rate of the universe. This implies that it takes longer for the expansion rate to fall to the observed value of H_0. For $\Omega_{\Lambda 0} = 0.7$ and $h = 0.65$, the age of the universe is 14 Gyr.

The variation of the age of the universe with the value of the cosmological constant is shown in Fig. (6.5). For $\Omega_{\Lambda 0} = 0.7$, as inferred from the supernovae observations, the age of the universe is enhanced by a factor of 1.45: $t_0 = 1.45(6.5/h)\,\mathrm{Gyr} = 9.4h^{-1}\,\mathrm{Gyr}$. So a value of $h = 0.65$ implies $t_0 = 14.5\,\mathrm{Gyr}$ and this is comfortably within the limits set by globular cluster ages.

6.6 The Fate of a Universe with a Cosmological Constant

We have seen that the recollapse of a pressureless universe is only possible for $kc^2 = +1$, but nevertheless, when this condition is satisfied, the recollapse is inevitable. Since a cosmological constant enhances the expansion, it is possible that it may prevent the recollapse if its numerical value is sufficiently high. Here we investigate this possibility further.

If the universe is ever to recollapse, the right–hand side of Eq. (6.24) must vanish at some instant in time. Since the density of matter is positive, $\rho > 0$, this can *never* happen if the inequality

$$\frac{8\pi G\Lambda}{3} > \frac{1}{a_0^2} \tag{6.38}$$

is satisfied – if this inequality is true today, it remains true for all future times since the scale factor is increasing with time. We can rewrite the condition (6.38) in terms of the Ω_Λ–parameter for the cosmological constant by dividing both sides by H_0^2. Thus, recollapse is not possible if

$$\Omega_{\Lambda 0} > \frac{1}{a_0^2 H_0^2} \tag{6.39}$$

However, the Friedman equation (6.26) implies that Eq. (6.39) is equivalent to the condition that

$$\Omega_{M0} < 1 \tag{6.40}$$

Hence, a closed universe with a cosmological constant will never recollapse if the density of ordinary, pressureless matter satisfies Eq. (6.40) and this is indeed what we observe today, $\Omega_{M0} \approx 0.3$. The full dependence of the universe's fate on the parameters Ω_Λ and Ω_M is shown in Fig. 6.6.

It appears, therefore, that the destiny of our universe is to expand forever regardless of the amount of ordinary matter that is present within it. As the density of matter decreases, the cosmological constant will come to dominate the universe and the universe will become effectively cold and empty.

This concludes our discussion on the future of the universe. In the remainder (second half) of the course we will trace the past history of the universe from its origin in the big bang through to the epoch when galaxies formed and hence to the present day.