

8 The Early Universe II

8.1 Neutrino Decoupling

Neutrinos are kept in equilibrium at high temperatures by reactions involving the weak nuclear force. To keep the analysis as simple as possible, consider the reaction where an electron and positron collide to form a neutrino/anti-neutrino pair:



At sufficiently high temperatures the reverse reaction is also possible. Hence the neutrinos are held in equilibrium. However, the cross-section for this reaction is sensitive to the temperature of the universe and is given by

$$\sigma = \frac{G_F^2 T^2}{c} \quad (8.2)$$

where G_F is a constant known as the Fermi constant. This reaction occurs when the reaction rate exceeds the Hubble time (see Section 7.3). However, condition (7.9) implies that for the reaction to proceed we must satisfy

$$\sigma cn > H \quad (8.3)$$

Since the number density of the particles scales as $n \propto a^{-3} \propto T^3$, the left-hand side of (8.3) falls as T^5 . On the other hand, the Friedmann equation implies that $H^2 \propto \rho \propto T^4$. Hence, the left hand side of Eq. (8.3) falls with temperature more rapidly than the right hand side. This implies that when the temperature falls below a certain critical value, it will no longer be possible to satisfy condition (8.3) and from that time on, the conversion of electrons and positrons into neutrinos (and vice-versa) ceases.

A full analysis indicates that the critical temperature is of the order of 10^{10} K. Now, once the neutrinos have decoupled they cease to interact with the matter and so should survive through to the present epoch. Since they were in thermal equilibrium, they should remain so at the present time. A fundamental prediction of the hot big bang model therefore is that the universe should be bathed in a neutrino background in much the same way that it is bathed in the cosmic microwave radiation¹⁸.

We might expect the temperature of the neutrinos should match that of the cosmic microwave background ($T_{\text{rad}} = 2.728$ K), since the energy density of both the neutrinos and photons scale as T^4 . However, there is a subtlety that makes the radiation temperature slightly higher. The subtlety is that when the neutrinos decouple – that is, when the reaction (8.1) ceases – the temperature is still above the limiting temperature for the electrons to be relativistic, i.e., $k_B T > m_e c^2 \approx 0.5$ MeV. (See

¹⁸Such a background of neutrinos has yet to be discovered and it remains difficult to see how such a detection could be made since neutrinos interact so weakly with matter.

Problem Sheet 6, Question 3). Hence, two photons can still collide and convert into an electron–positron pair, as outlined in Section 7.4:



This reaction is possible as long as the temperature exceeds 1 MeV , i.e., 6×10^9 K. When the temperature falls to this critical value, the photons lack sufficient energy to convert into mass and the result is that the electrons and positrons rapidly annihilate into photons. Consequently, the number of photons rapidly increases. Indeed, conservation of number densities implies that the total number density of electrons, positrons and photons before annihilation must equal the number density of photons afterwards:

$$(n_{e^-} + n_{e^+} + n_\gamma)_{\text{before}} = n_\gamma|_{\text{after}} \quad (8.5)$$

The crucial point is that the temperature of the photons before the electrons and positrons annihilate is equal to that of the neutrinos. On the other hand, the photons receive extra energy as the electrons and positrons annihilate, whereas the neutrinos do not because they are already decoupled. Furthermore, as we saw in the previous subsection, a fermion at a temperature T may be viewed as a boson with a corrected degeneracy factor. Such a result applies here with respect to the number density of the electrons. Integrating Eq. (7.12) yields the number density of relativistic electrons as

$$n_e = \frac{4\pi g_e}{h^3} \int_0^\infty \frac{p^2 dp}{e^{pc/k_B T} + 1} \quad (8.6)$$

and due to the identity (7.16) we may rewrite this as

$$\begin{aligned} n_e &= \frac{4\pi g_e}{h^3} \left[\int_0^\infty \frac{p^2 dp}{e^{pc/k_B T} - 1} - 2 \int_0^\infty \frac{p^2 dp}{e^{2pc/k_B T} - 1} \right] \\ &= \frac{4\pi g_e}{h^3} \left[1 - \frac{1}{4} \right] \int_0^\infty \frac{p^2 dp}{e^{pc/k_B T} - 1} \end{aligned} \quad (8.7)$$

where we make the substitution $q = 2p$ in the second integral on the first line of this expression.

In other words, remembering that electrons and photons have the same degeneracy factor of 2, the number density of the electrons is just 3/4 that of the photons:

$$n_e = \frac{3}{4} n_\gamma \quad (8.8)$$

Hence¹⁹

$$\left(\frac{3}{4} + \frac{3}{4} + 1 \right) n_\gamma|_{\text{before}} = \frac{10}{4} n_\gamma|_{\text{before}} = n_\gamma|_{\text{after}} \quad (8.9)$$

¹⁹Actually, the proper way to calculate the temperature is in terms of entropy conservation and this gives a factor of 11/4 rather than 10/4. However, as we have not looked at entropy in this course, we would need to do extra work to proceed along such lines. The calculation outlined here suffices.

It follows, therefore, that since $n \propto T^3$ that

$$\frac{10}{4}T_\gamma^3|_{\text{before}} = \frac{10}{4}T_\nu^3|_{\text{before}} = T_\gamma^3|_{\text{after}} \quad (8.10)$$

Since the neutrino and photon temperatures both scale after this event as $1/a$, we may conclude that the present-day temperature of the neutrino background should be

$$T_{\nu,0} = \left(\frac{4}{10}\right)^{1/3} T_{\text{rad},0} = 2 \text{ K} \quad (8.11)$$

The detection of such a background would be a major triumph for the big bang model.

8.2 Photon to Baryon Ratio

We have seen that our observations indicate that the universe is (probably) dominated by a cosmological constant today, making up 70% of the density, with the remaining 30% comprised of ordinary pressureless matter. There is also the cosmic microwave background (CMB) radiation. This radiation appears to have the same properties regardless of the direction it comes from. Its spectrum is that of a (thermal) black body distribution with a temperature $T = 2.728 \pm 0.004 \text{ K}$. The peak of the spectrum is in the microwave wavelength range.

We have neglected the effective density of this radiation in our discussions thus far. We now justify this by calculating the effective mass density of the radiation relative to the critical density at the present epoch, i.e., $\Omega_{\text{rad}} = \rho_{\text{rad}}/\rho_c$.

Recall from Eq. (3.15) that the *energy* density of radiation at a temperature T is given by

$$\epsilon_{\text{rad}} = \alpha T^4 \quad (8.12)$$

where the constant of proportionality

$$\alpha = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3} = 7.565 \times 10^{-16} \text{ J m}^{-3} \text{ K}^{-4} \quad (8.13)$$

is known as the radiation constant. For the observed temperature of the CMB this implies that

$$\epsilon_{\text{rad},0} = 4.19 \times 10^{-14} \text{ J m}^{-3} \quad (8.14)$$

A given energy density can be related to a mass density as a consequence of Einstein's relation $E = mc^2$. Hence,

$$\rho_{\text{rad},0} = \frac{\epsilon_{\text{rad},0}}{c^2} = 4.66 \times 10^{-31} \text{ kg m}^{-3} \quad (8.15)$$

where we make use of the fact that $1 \text{ J} = 1 \text{ kg m}^2 \text{ sec}^{-2}$ (see Appendix A). Converting this into the fraction of the total density then implies that

$$\Omega_{\text{rad},0} = \frac{\rho_{\text{rad},0}}{\rho_{\text{crit},0}} = 2.47 \times 10^{-5} h^{-2} \quad (8.16)$$

where we have employed the expression (4.1) for the critical density. Thus, the fraction of matter in the universe in microwave radiation today is indeed small when compared to that of the matter, $\Omega_{M0} \approx 0.3$, and we are justified in neglecting it in earlier Sections.

Now that we have introduced the idea that the radiation can be thought of in terms of particles (photons), an interesting (and important) question to answer is how many photons are there in the universe compared to the number of baryons? We denote the number density of photons by n_γ and the number density of baryons (protons and neutrons) by n_B . Since density scales as the inverse of volume as the universe expands, $n_\gamma/n_B = \text{constant}$. This is important because if we can determine this ratio today, we can determine it for all times and, in particular, at times when the universe was very much smaller and hotter than it is at present.

The mean energy of a photon in a blackbody distribution is (see Section 3.2)

$$\langle E \rangle \approx 3k_B T \quad (8.17)$$

For a temperature $T = 2.728 \text{ K}$, this implies that $\langle E \rangle \approx 1.13 \times 10^{-22} \text{ J}$. Thus, the number density of photons in the universe today is

$$n_{\gamma,0} = \frac{\epsilon_{\text{rad}}}{\langle E \rangle} \approx \frac{4.19 \times 10^{-14}}{1.13 \times 10^{-22}} \text{ m}^{-3} = 3.7 \times 10^8 \text{ m}^{-3} \quad (8.18)$$

In other words, at least in outer space, there are almost a billion (10^9) photons per cubic metre. How does this compare with the number density of baryons? We shall see in the lectures on primordial nucleosynthesis that the baryons in the universe contribute a fraction $\Omega_B = \rho_B/\rho_c \approx 0.02h^{-2}$ to the total density at the present epoch. Converting this to an energy density

$$\epsilon_{B0} = \rho_B c^2 = \rho_c \Omega_B c^2 = 3.38 \times 10^{-11} \text{ J m}^{-3} \quad (8.19)$$

Assuming a mass–energy of $m_P c^2 \approx 939 \text{ MeV} \approx 1.5 \times 10^{-10} \text{ J}$ for baryons then implies that

$$n_{B0} = \frac{\epsilon_B}{m_P c^2} \approx 0.22 \text{ m}^{-3} \quad (8.20)$$

and we deduce that

$$\frac{n_\gamma}{n_B} \approx 1.7 \times 10^9 \quad (8.21)$$

This implies that there are 1.7×10^9 photons for every baryon in the universe. This is valid today and in the very early universe since the ratio is time–independent. The photons dominate numerically but most of the density comes from the baryons.

8.3 The Origin of the Microwave Background

We can now explain the origin of the cosmic microwave background. The key point is that above a critical temperature, a hydrogen atom becomes *ionized*. Specifically,

the ionization energy of the hydrogen atom is $E_{\text{ion}} = 13.6 \text{ eV}$ – it takes this amount of energy to excite the electron in its ground state sufficiently for it to escape from the nucleus of the atom. At sufficiently high temperatures (early times) the energy of the photons in the universe exceeded this critical value and the hydrogen was therefore ionized. Let us consider some early era when the universe consisted of a hot, ionized plasma of free nuclei, electrons and photons. The radiation interacted primarily with the free electrons through Thompson scattering – the process where X-rays scatter from free electrons.

As the universe expanded, its temperature dropped. Eventually the photons lacked sufficient energy to keep the hydrogen ionized. The electrons were therefore able to combine with the nuclei with the result that no naked electric charge remained in the universe. Consequently, the photons were no longer able to interact directly with the matter and were able to propagate unhindered. The temperature when this occurred is known as the *decoupling temperature*, T_{dec} , since the matter and radiation effectively decoupled from one another at this time. The radiation has remained essentially undisturbed through to the present day, although it has lost energy due to the expansion of the universe. It is this radiation that we identify with the cosmic microwave background.

The mean energy of a photon in a blackbody distribution is given by Eq. (8.17), $\langle E \rangle \approx 3k_B T$. We might think, therefore, that when the mean energy of photons fell below the hydrogen ionization energy, the electrons were able to bind to the nuclei. This is not so however. The form of the blackbody distribution implies that a small fraction of the photon distribution has an energy much greater than the average (see the Figure of the microwave background spectrum and Eq. (3.12)). This fraction is exponentially suppressed and specifically the fraction of photons exceeding a given energy I at a temperature T is given by $\exp[-I/(k_B T)]$. Thus, the number density of photons, n_{ion} , capable of ionizing a hydrogen atom at temperature T is

$$\frac{n_{\text{ion}}}{n_\gamma} = \exp\left[-\frac{E_{\text{ion}}}{k_B T}\right] \quad (8.22)$$

where $E_{\text{ion}} = 13.6 \text{ eV}$. Consequently, even after the mean photon energy fell below the hydrogen ionization energy, there were still some photons with an energy in excess of E_{ion} .

Now, there are many more photons in the universe than there are electrons. Since the universe is electrically neutral, the number density of electrons is the same as that of the protons, so $n_e = n_B \approx 10^{-9} n_\gamma$, as follows from Eq. (8.21). Thus, we may rewrite Eq. (8.22) as

$$\frac{n_{\text{ion}}}{n_e} = \frac{n_\gamma}{n_B} \exp\left[-\frac{E_{\text{ion}}}{k_B T}\right] \approx 10^9 \exp\left[-\frac{E_{\text{ion}}}{k_B T}\right] \quad (8.23)$$

If we need just one photon to have sufficient energy to ionize a hydrogen atom, then a neutral hydrogen atom can not form until $n_{\text{ion}} < n_e$ and from Eq. (8.23) this implies

that neutral hydrogen forms when the temperature has fallen to

$$T_{\text{dec}} \approx \frac{13.6}{k_B \ln(10^9)} \approx 7500 \text{ K} \quad (8.24)$$

This calculation is not precisely accurate, but all of the essential physics is there. (It turns out that there are additional corrections to the exponential suppression factor in Eq. (8.22), but we would need another lecture to consider these in detail. The above analysis suffices to highlight the important physics). The full calculation implies that the decoupling temperature is

$$T_{\text{dec}} = 3000 \text{ K} \quad (8.25)$$

This implies that the matter and radiation decoupling when the universe was one thousandth its present size, i.e., at a redshift of $z_{\text{dec}} \approx 1000$.

After decoupling, the matter and radiation evolve independently. Since the temperature of the radiation falls inversely with the scale factor according to Eq. (7.1), the thermal distribution of the radiation also evolves. However, the blackbody form of the spectrum is preserved. Returning to Eq. (3.12), we know that the temperature falls as a^{-1} but so does the frequency, f , since $f \propto \lambda^{-1} \propto a^{-1}$. Hence, the form of the denominator on the right hand side of Eq. (3.12) is *independent* of the scale factor. For a given frequency interval, the numerator falls as the inverse of the volume of the universe, $f^3 \propto a^{-3}$, and this is precisely how a density should fall.

As a result, after decoupling has been completed, the blackbody form of the spectrum is preserved, but its temperature falls inversely with the scale factor. Indeed, the radiation remains in a thermal distribution through to the present epoch. This is a very important conclusion. It implies that if thermal equilibrium was established at some early time, then the thermal distribution survives through to later times. The origin of the cosmic microwave radiation that we observe today is therefore explained in the hot big bang model. This radiation is the radiation that is left over from the early universe when the universe was so hot that neutral atoms could not form and the charged matter and radiation interacted sufficiently frequently for thermal equilibrium to be established. *The reason why the microwave background is a black body is that it was once in thermal equilibrium with the matter when the universe was much younger and hotter than it is today.*

8.4 Relationships between Temperature and Age of the Universe

We can now proceed to derive some important expressions that relate the temperature of the universe to its age.

Firstly, we should note that in the arena of the early universe, it is consistent to neglect the effects of the curvature term, kc^2/a^2 , in the Friedmann equation. This

follows since the densities of relativistic and non-relativistic particles scale as $\rho_{\text{rel}} \propto a^{-4}$ and $\rho_{\text{nonrel}} \propto a^{-3}$, respectively. Thus, as $a \rightarrow 0$, the densities diverge more rapidly than the curvature term. (An alternative way of seeing this is provided in the answer to question 4 of Exercise Sheet 6).

Now, we can view the universe today as a mixture of relativistic and non-relativistic particles, or equivalently, as radiation and pressureless matter. Taking the ratio of their respective densities implies that

$$\frac{\rho_{\text{rel}}}{\rho_{\text{nonrel}}} = \frac{\Omega_{\text{rel}}}{\Omega_{\text{nonrel}}} \propto \frac{1}{a} \quad (8.26)$$

and fixing values to the present-day, we deduce that

$$\frac{\Omega_{\text{rel}}}{\Omega_{\text{nonrel}}} = \left(\frac{\Omega_{\text{rel}}}{\Omega_{\text{nonrel}}} \right)_0 \frac{a_0}{a} \quad (8.27)$$

We have seen that today that the density of radiation is $\Omega_{\text{rad},0} = 2.47 \times 10^{-5} h^{-2}$. However, we should also account for the contribution of the three neutrino species, Ω_{ν} . We would expect the contribution of the neutrinos to be similar. As is outlined in Questions 1 and 2 of Exercise Sheet 7 and Section 8.1, this is indeed the case, but there is a slight difference in the numerical value. It turns out that the present temperature of the neutrinos is slightly lower, at $T_{\nu 0} = 2$ K, than that of the microwave radiation. This implies that the density of neutrinos is also slightly lower and it can be shown that $\Omega_{\nu 0} = 0.68 \Omega_{\text{rad} 0}$. Thus, the total fractional contribution of relativistic particles today is

$$\Omega_{\text{rel,tot},0} = 4.17 \times 10^{-5} h^{-2} \quad (8.28)$$

In what follows we are going to assume that the cosmological constant is zero. One reason for this is that although there is evidence, as we have seen in Section 6, for a non-zero cosmological constant, this is by no means universally accepted at present. Secondly, if such a term really is present, it will not significantly affect the forthcoming conclusions, because it has only just started to dominate at the present epoch. Thirdly, and by no means least, the equations have a much simpler form if we neglect this term!

In the absence of a cosmological constant, the current density of non-relativistic matter is just Ω_0 . Thus, substituting this and Eq. (8.28) into Eq. (8.27) implies that the ratio of densities of relativistic to non-relativistic matter is related to the size of the universe by

$$\frac{\Omega_{\text{rel}}}{\Omega_{\text{nonrel}}} = \frac{4.17 \times 10^{-5} a_0}{\Omega_0 h^2} \frac{a_0}{a} \quad (8.29)$$

This is an important equation because it enables us to compute the relative amounts of the different types of matter present in the universe at different eras. For example, we have seen that the matter and radiation decoupled when the universe was one thousandth its present size, i.e., $a = 10^{-3} a_0$. Thus, at that time,

$$\frac{\Omega_{\text{rel}}}{\Omega_{\text{nonrel}}} \approx \frac{0.04}{\Omega_0 h^2} \quad (8.30)$$

Since, roughly speaking, $\Omega_0 h^2 = \mathcal{O}(1)$, this implies that non-relativistic matter dominated the radiation at decoupling. There existed an earlier time, known as the *epoch of matter–radiation equality*, and denoted t_{eq} , when the densities were equal, $\Omega_{\text{rel}} = \Omega_{\text{nonrel}}$. We find from Eq. (8.29) that the scale factor at that time was

$$a(t_{\text{eq}}) = a_{\text{eq}} = \frac{1}{24000\Omega_0 h^2} a_0 \quad (8.31)$$

Thus, when $a > a_{\text{eq}}$, non-relativistic matter dominates the universe and for $a < a_{\text{eq}}$, relativistic matter dominates. We say that the universe is radiation dominated for $t < t_{\text{eq}}$ and matter dominated for $t > t_{\text{eq}}$. This provides a more heuristic estimate of the epoch of matter–radiation equality than that deduced from the solution to the Friedmann equation we derived in Section (3.4).

We now calculate when this transition between matter and radiation domination occurred. The first point to note is that in the early universe, the curvature term k/a^2 in the Friedmann equation becomes negligible relative to the matter and radiation since their densities vary as a^{-3} and a^{-4} , respectively, and therefore grow more rapidly as $a \rightarrow 0$. Thus, it is a good approximation to neglect the curvature term in the Friedmann equation at early times. (See Problem Sheet 3, Question 1). Consequently, let us return to the Friedmann equation (2.27) for $k = 0$ when the universe is radiation-dominated:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G \rho_{\text{eq}} a_{\text{eq}}^4}{3a^4} \quad (8.32)$$

where we have normalized the radiation density at the epoch of matter–radiation equality. For simplicity, we also neglect the small numerical correction that arises by considering the different weighting of fermion and boson particles. This does not affect the argument (see Question 2, Exercise Sheet 6).

Integrating and converting to temperature via Eq. (3.15) (and remembering to convert from energy density to mass density) implies that

$$T = T_{\text{eq}} \left(\frac{32\pi G \rho_{\text{eq}}}{3} \right)^{-1/4} \frac{1}{t^{1/2}} \quad (8.33)$$

However, from Eq. (3.15), the density of radiation at a temperature T_{eq} is $\rho_{\text{eq}} = \alpha T_{\text{eq}}^4 / c^2$. Substituting into Eq. (8.33) implies that

$$T = \left(\frac{32\pi G \alpha}{3c^2} \right)^{-1/4} \frac{1}{t^{1/2}} = \left(\frac{32\pi^3 G k_B^4}{45 \hbar^3 c^5} \right)^{-1/4} \frac{1}{t^{1/2}} \quad \text{if } T > T_{\text{eq}} \quad (8.34)$$

Eq. (8.34) is another key relation, because it measures the temperature of the universe (before the epoch of matter–radiation equality) directly to its age. We refer to it as the **Temperature–Time Relation**. The precise relationship is determined

in terms of the four fundamental constants of nature, $\{G, c, \hbar, k_B\}$ and numerically,

$$\begin{aligned} T &= 1.5 \times 10^{10} \left(\frac{t}{\text{sec}} \right)^{-1/2} \text{ K} \\ k_B T &= 1.3 \left(\frac{t}{\text{sec}} \right)^{-1/2} \text{ MeV} \end{aligned} \quad (8.35)$$

We see then that when the universe was about one second old, the temperature was of the order 10^{10} K, comparable to the temperature of the sun's core. In particular nuclear reactions occur for $T > 10^9$ K, corresponding to $t < 400$ s. The maximum energy scales that the world's most powerful particle accelerators can probe are 1 TeV, corresponding to $t \approx 10^{-12}$ s. In some sense, these accelerators probe the physics that operated in the universe when it was at that age. Any earlier epoch can not be directly probed by earth-based experiments. We shall see after the Sections on inflation that cosmology may provide a unique window to higher energy scales and earlier times.

It is important to emphasize that Eq. (8.34) holds for $T > T_{\text{eq}}$. For temperatures below this critical value, the universe is dominated by pressureless matter and we have $T \propto a^{-1} \propto t^{-2/3}$, where again we neglect the small difference that arises in the time dependence due to spatial curvature. In this case, we may normalize at the present epoch, $T = T_0(t_0/t)^{2/3}$, where the present time t_0 denotes the age of the universe. Taking the value (5.47), $t_0 = 6.5h^{-1}$ Gyr, for the age and $T_0 = 2.7$ K for the present temperature then implies that

$$T = 9.4 \times 10^{11} h^{-2/3} \left(\frac{t}{\text{sec}} \right)^{-2/3} \text{ K} \quad \text{if } T < T_{\text{eq}} \quad (8.36)$$

Moreover, since $a_{\text{eq}}/a_0 = (t_{\text{eq}}/t_0)^{2/3}$, it follows from Eq. (8.31) that the epoch of matter–radiation equality occurs at a time

$$t_{\text{eq}} = 2.7 \times 10^{-7} \Omega_0^{-3/2} h^{-3} t_0 = 1700 \Omega_0^{-3/2} h^{-4} \text{ yr} \quad (8.37)$$

and from Eq. (8.36), this occurs at a temperature

$$T_{\text{eq}} = 66,000 \Omega_0 h^2 \text{ K} \quad (8.38)$$

It is interesting to note that the temperature and age of the universe at matter–radiation equality is sensitive to the two key cosmological parameters Ω_0 and h (the reduced Hubble constant). A precise measurement of these parameters then determines the numerical value of the temperature at that time. The temperature T_{eq} is much higher than the decoupling temperature (8.25) of 3000 K and indeed we deduce from Eq. (8.36) that decoupling occurs when the age of the universe is

$$t_{\text{dec}} = 175,000 h^{-1} \text{ yrs} \quad (8.39)$$

Thus, the epoch of decoupling occurs *after* the epoch of matter–radiation equality. To summarize this Section, therefore, we have seen then that as we go back in time

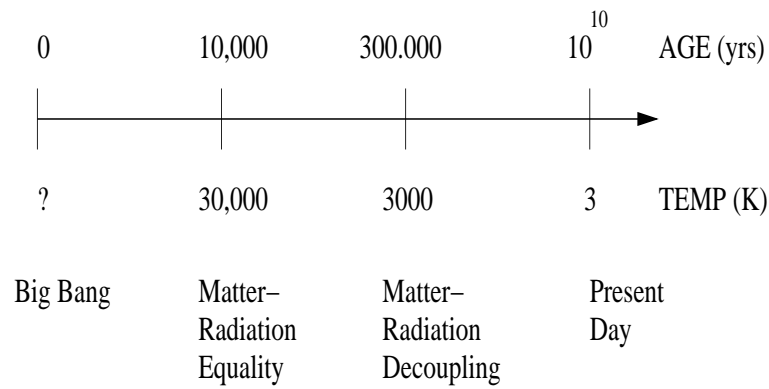


Fig. (8.1). A schematic plot comparing the timescales and epochs of matter-radiation equality and matter-radiation decoupling to the present epoch. Note how the factor of three appears in the numerical estimates.

the matter and radiation are coupled during the first 300,000 years of the universe's existence (for $h \approx 0.65$) and the universe becomes effectively matter dominated some 6000 years or so after the big bang. It is dominated by radiation and relativistic particles before this time. A timeline outlining these significant events in the history of the universe is shown in Fig. (8.1). The next epoch of interest occurs a few minutes after the big bang – this is the epoch of primordial nucleosynthesis and the subject of the next Section.