Part II, Gravity as a space-time geometry

We know from Part I that the fundamental physical concept of General Relativity is that gravitational field is identical to geometry of curved space-time. In this Part II we will see how this idea, called the Geometrical principle, entirely determines the mathematical structure of General Relativity.

1. The Principle of Equivalence in Newtonian Gravity

As we already know from Part I, all bodies in given gravitational field move in the same manner, if initial conditions are the same. In other words, in given gravitational field all bodies move with the same acceleration. In absence of gravitational Field, all bodies move also with the same acceleration relative to the non-inertial frame. Thus we can formulate the Principle of Equivalence which says: Locally, any non-inertial frame of reference is equivalent to a certain gravitational field.

The important consequence of the Principle of Equivalence is that locally gravitational field can be eliminated by proper choice of the frame of reference. Such frames of reference are called locally inertial or galilean frames of reference.

Globally (not locally), "actual" Gravitational Fields can be distinguished from corresponding non-inertial frame of reference by its behavior at infinity: Gravitational Fields generated by gravitating bodies decay with distance.

In Newton theory the motion of a test particle is determined by the following equation of motion

$$m_{in}\vec{a} = -m_{gr}\nabla\phi,$$

where $\vec{a}$ is the acceleration of the test particle, ϕ is newtonian potential of gravitational field, m_{in} is the inertial mass of the test particle and m_{gr} is its gravitational mass, which is the gravitational analog of the electric charge in the theory of electromagnetism. The fundamental property of gravitational fields that all test particles move with the same acceleration for given ϕ is explained within frame of newtonian theory just by the following "coincidence":

$$\frac{m_{in}}{m_g} = 1,$$

i.e. inertial mass m_{in} is equal to gravitational mass m_{gr}.

2. The Principle of Equivalence in General Relativity

As it is known from every course on Special Relativity, this theory works only in frames of reference of the special kind called Global Inertial Frames of Reference. For such frames of reference the following combination of time and space coordinates remains invariant whatever global inertial frame of references is chosen

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

This combination is called the interval. As it is known from every course on Special Relativity, all space-time coordinates in different global inertial frames of reference are related with each other by the Lorentz transformations, it is known also that these transformations leave the shape of the interval unchanged. But this is not the case if one considers transformation of coordinates in more general case, when at least one of frames of reference is non-inertial. This interval is not reduced anymore to the simple sum of squares of the coordinate differentials and can be written in the following more general quadratic form:

$$ds^2 = g_{ik}dx^i dx^k \equiv \sum_{i=0}^{3}\sum_{k=0}^{3} g_{ik}dx^i dx^k,$$

where repeating indices mean summation. In inertial frames of reference

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1,$$

and

$$g_{ik} = 0, \quad \text{if} \quad i \neq k.$$

Example. Transformation to uniformly rotating frame.

$$x = x' \cos \Omega t - y' \sin \Omega t,$$
$$y = x' \sin \Omega t + y' \cos \Omega t,$$
$$z = z',$$

where Ω is the angular velocity of rotation around z-axis. In this non-inertial frame of reference

$$ds^2 = [c^2 - \Omega^2(x'^2 + y'^2)]dt^2 - dx'^2 - dy'^2 - dz'^2 +$$
$$+2\Omega y' dx' dt - 2\Omega x' dy' dt.$$

According to General Theory of Relativity gravity is nothing but manifestation of space-time 4-geometry, this geometry is determined by by metric

$$ds^2 = g_{ik}(x^m)dx^i dx^k,$$

where $g_{ik}(x^m)$ is called the metric tensor. What is exactly meant by the term "tensor" we will discuss in the next lecture. At the present moment we can consider $g_{ik}(x^m)$ as a 4×4 -matrix and all its components in general case can depend on all 4 coordinates x^m, where $m = 0, 1, 2, 3$. All information about the geometry of space-time is contained in $g_{ik}(x^m)$.The dependence of $g_{ik}(x^m)$ on x^m means that this geometry is different in different events, which implies that the space-time is curved and its geometry is not Euclidian. Such sort of geometry is the the subject of mathematical discipline called Differential Geometry developed in XIX Century.

The General Relativity gives very simple and natural explanation of the Principle of Equivalence: In curved space-time all bodies move along geodesics, that is why their world lines are the same in given gravitational field. The situation is the same as in flat space-time when free particles move along straight lines which are geodesics in flat space-time. What is geodesic we will also discuss the next lecture.

The one of the main statements of General Theory of Relativity is the following: If we know g_{ik}, we can determine completely the motion of test particles and performance of all test fields. [Test particle or test field means that gravitational field generated by these test objects is negligible.] In the next lecture we will see that the metric tensor g_{ik} itself is determined by physical content of the space-time.

In any curved space-time (i.e in the actual gravitational field)there is no global galilean frames of reference. In flat space-time, if me work in non-inertial frames of reference metrics looks like the metric in gravitational field (because according to the Equivalence Principle locally actual gravitational field is not distinguishable from corresponding non-inertial frame of reference), nevertheless local galilean frames of reference do exist. The local galilean frame of reference is equivalent to the freely falling frame of reference in which locally gravitational field is eliminated. From geometrical point of view to eliminate gravitational field locally means to find such frame of reference in which

$$g_{ik} \rightarrow \eta_{ik} \equiv \mathrm{diag}(1, -1, -1, -1).$$

3. The Principle of Covariance

This principle says: **The shape of all physical equations should be the same in an arbitrary frame of reference**, **including the most general case of non-inertial frames (in contrast to the Special Theory of Relativity which works only in inertial frames of reference).**
This principle predetermines the mathematical structure of General Relativity: All equations should contain only tensors. By definition, tensors are objects which are transformed properly in the course of coordinate transformations from one frame of reference to another.
Taking into account that non-inertial frames of reference in 4-dimensional space-time correspond to curvilinear coordinates, it is necessary to develop four-dimensional differential geometry in arbitrary curvilinear coordinates.