Mathematical Structure of General Relativity

We know from the Parts I that the fundamental physical concept of General Relativity is that gravitational field is identical to geometry of curved space-time. In the Part II we discussed how this idea, called the Geometrical principle, entirely determined the mathematical structure of General Relativity. We have finished with the formulation of the Covariance Principle. This lecture is about mathematical realization of this principle in General Relativity.

In these notes you will find some material which I just mentioned in the lecture to save time (see pages 5-8 and 11-12). However, this material is really very important for understanding General Relativity. I recommend you to read it carefully. I hope that Course Work 3 will help you in your reading.

1. Curvilinear coordinates

Let us consider the transformation of coordinates from one frame of reference (x^0, x^1, x^2, x^3) **to another,**$(x^{'0}, x^{'1}, x^{'2}, x^{'3})$:

$$x^0 = f^0(x'^0, x'^1, x'^2, x'^3),$$
$$x^1 = f^1(x'^0, x'^1, x'^2, x'^3),$$
$$x^2 = f^2(x'^0, x'^1, x'^2, x'^3),$$
$$x^3 = f^1(x'^0, x'^1, x'^2, x'^3).$$

Then

$$dx^i = \frac{\partial x^i}{\partial x'^k} dx'^k = S^i_k dx'^k, \quad i, k = 0,\ 1,\ 2,\ 3,$$

where

$$S^i_k = \frac{\partial x^i}{\partial x'^k}$$

is a transformation matrix. Remember that all repeating indices mean summation, otherwise even such basic transformation would be written ugly. To demonstrate that summation convention is really very useful, I will write, the first and the last time, the same transformation without using the summation convention

$$dx^0 = \frac{\partial x^0}{\partial x'^0} dx'^0 + \frac{\partial x^0}{\partial x'^1} dx'^1 + \frac{\partial x^0}{\partial x'^2} dx'^2 + \frac{\partial x^0}{\partial x'^3} dx'^3 =$$
$$= S^0_0 dx'^0 + S^0_1 dx'^1 + S^0_2 dx'^2 + S^0_3 dx'^3,$$
$$dx^1 = \frac{\partial x^1}{\partial x'^0} dx'^0 + \frac{\partial x^1}{\partial x'^1} dx'^1 + \frac{\partial x^1}{\partial x'^2} dx'^2 + \frac{\partial x^1}{\partial x'^3} dx'^3 =$$
$$= S^1_0 dx'^0 + S^1_1 dx'^1 + S^1_2 dx'^2 + S^1_3 dx'^3,$$
$$dx^2 = \frac{\partial x^2}{\partial x'^0} dx'^0 + \frac{\partial x^2}{\partial x'^1} dx'^1 + \frac{\partial x^2}{\partial x'^2} dx'^2 + \frac{\partial x^2}{\partial x'^3} dx'^3 =$$
$$= S^2_0 dx'^0 + S^2_1 dx'^1 + S^2_2 dx'^2 + S^2_3 dx'^3,$$
$$dx^3 = \frac{\partial x^3}{\partial x'^0} dx'^0 + \frac{\partial x^3}{\partial x'^1} dx'^1 + \frac{\partial x^3}{\partial x'^2} dx'^2 + \frac{\partial x^3}{\partial x'^3} dx'^3 =$$
$$= S^3_0 dx'^0 + S^3_1 dx'^1 + S^3_2 dx'^2 + S^3_3 dx'^3.$$

2.Tensors

Now we can give the definition of the Contravariant four-vector:
The Contravariant four-vector is the combination of four quantities (components) A^i, which are transformed like differentials of coordinates:

$$A^i = S^i_k A'^k.$$

Let φ is scalar field, then

$$\frac{\partial \varphi}{\partial x^i} = \frac{\partial \varphi}{\partial x'^k}\frac{\partial x'^k}{\partial x^i} = \tilde{S}^k_i \frac{\partial \varphi}{\partial x'^k},$$

where $\tilde{S}^k_i$ is another transformation matric. What is relation of this matrix with the previous transformation matrix S^i_k? If we take product of these matrices, we obtain

$$S^i_n \tilde{S}^n_k = \frac{\partial x^i}{\partial x'^n}\frac{\partial x'^n}{\partial x^k} = \frac{\partial x^i}{\partial x^k} = \delta^i_k,$$

where δ^i_k is so called Kronneker symbol, which actually is nothing but the unit matrix:

$$\delta^i_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In other words $\tilde{S}^i_k$ is inverse or reciprocal with respect to S^i_k.

Now we can give the definition of the Covariant four-vector:
The Covariant four-vector is the combination of four quantities (components) A_i, which are transformed like like components of the gradient of a scalar field:

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k.$$

Note, that for contravariant vectors we always use upper indices, which are called contravariant indices, while for covariant vectors we use low indices, which are called covariant indices. In General Relativity summation convention always means that on of the two repeating indices should be contravariant and another should be covariant. For example,

$$A^i B_i = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \ \textbf{(scalar product)},$$

while there is no summation if both indices are, say, covariant:

$$A_i B_i = \begin{matrix} A_0 B_0, & \text{if } i = 0 \\ A_1 B_1, & \text{if } i = 1 \\ A_2 B_2, & \text{if } i = 2 \\ A_3 B_3, & \text{if } i = 3 \end{matrix}$$

Now we can generalize the definitions of vectors and introduce tensors entirely in terms of transformation laws.

Scalar is the tensor of the 0 rank. It has only $4^0 = 1$ component and 0 number of indices. Transformation law is

$$A = A',$$

we see that transformation matrices appear in transformation law 0 times.

Contravariant and covariant vectors are tensors of the 1 rank. They have $4^1 = 4$ components and 1 index. Corresponding transformation laws are

$$A^i = S^i_n A'^n,$$
$$A_i = \tilde{S}^n_i A'_n,$$

we see only 1 transformation matrix in each transformation law.

Contravariant tensor of the 2 rank has $4^2 = 16$ components and 2 contravariant indices. Corresponding transformation law is

$$A^{ik} = S^i_n S^k_m A'^{nm},$$

we see 2 transformation matrices in the transformation law.

Covariant tensor of the 2 rank has $4^2 = 16$ components and 2 covariant indices. Corresponding transformation law is

$$A_{ik} = \tilde{S}^n_i \tilde{S}^m_k A'_{nm},$$

we see 2 transformation matrices in the transformation law.

Mixed tensor of the 2 rank has $4^2 = 16$ components and 2 indices, 1 contravariant and 1 covariant. Corresponding transformation law is

$$A^i_k = S^i_n \tilde{S}^m_k A'^n_m,$$

we see 2 transformation matrices in the transformation law.

Covariant tensor of the 3 rank has $4^3 = 46$ components and 3 covariant indices. Corresponding transformation law is...

Mixed tensor of the $N + M$ rank with N contravariant and M covariant indices, has $4^{N+M} = 2^{2(N+M)}$ components and $N + M$ indices. Corresponding transformation law is

$$A^{i_1\, i_2\, \dots\, i_n}_{k_1\, k_2\, \dots\, k_M} = S^{i_1}_{n_1} S^{i_2}_{n_2} \dots S^{i_N}_{n_N} \tilde{S}^{m_1}_{k_1} \tilde{S}^{m_2}_{k_2} \dots \tilde{S}^{m_M}_{k_M} A'^{n_1\, n_2\, \dots\, n_N}_{m_1 m_2 \dots m_M},$$

we see N+M transformation matrices in the transformation law.

Reciprocal tensors

Two tensors A_{ik} and B^{ik} are called reciprocal to each other if

$$A_{ik}B^{kl} = \delta_i^l.$$

We can introduce now a contravariant metric tensor g^{ik} which is reciprocal to the covariant metric tensor g_{ik}:

$$g_{ik}g^{kl} = \delta_i^l.$$

With the help of the metric tensor and its reciprocal we can form contravariant tensor from covariant tensors and vice versa, for example:

$$A^i = g^{ik}A_k, \quad A_i = g_{ik}A^k,$$

in other words we can rise and descend indices as we like, some sort of juggling with indices. We can say that contravariant, covariant and mixed tensors can be considered as different representations of the same geometrical object.

For the contravariant metric tensor itself we have very important representation in terms of the transformation matrix from locally inertial frame of reference (galilean frame) to an arbitrary non-inertial frame, let us denote it as $S^i_{(0)k}$. We know that in the galilean frame of reference

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \eta^{ik} \equiv \mathrm{diag}(1, -1, -1, -1),$$

hence

$$g^{ik} = S^i_{(0)n}S^k_{(0)m}\eta^{lm} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3}.$$

This means that if we know the transformation law from the local galilean frame of reference to an arbitrary frame of reference, we know the metric at this arbitrary frame of reference and, hence, gravitational field!

Examples

1. Is the interval a scalar ?

Given that g_{ik} is a covariant tensor of the second rank and that

$$ds^2 = g_{ik}dx^i dx^k.$$

Prove that ds is a scalar.

The proof:

$$ds^2 = g_{ik}dx^i dx^k = (\tilde{S}_i^n \tilde{S}_k^m g'_{nm})(S_p^i dx'^p)(S_w^k dx'^w) = (\tilde{S}_i^n S_p^i)(\tilde{S}_k^m S_w^k)(g'_{nm}dx'^p dx'^w) =$$

$$= \delta_p^n \delta_w^m (g'_{nm}dx'^p dx'^w) = g'_{pw}dx'^p dx'^w = g'_{ik}dx'^i dx'^k = ds'^2,$$

hence $ds = ds'$ which means that ds is a scalar.

2. How many independent components in the metric tensor?

First, we can prove that the metric tensor is symmetric, i.e.

$$g_{ik} = g_{ki}.$$

Proof

$$ds^2 = g_{ik}dx^i dx^k = \frac{1}{2}(g_{ik}dx^i dx^k + g_{ik}dx^i dx^k) = \frac{1}{2}(g_{ki}dx^k dx^i + g_{ik}dx^i dx^k) = \frac{1}{2}(g_{ki} + g_{ik})dx^i dx^k =$$

$$= \tilde{g}_{ik}dx^i dx^k,$$

where

$$\tilde{g}_{ik} = \frac{1}{2}(g_{ki} + g_{ik}),$$

which is obviously symmetric one. Then we just drop "". The end of proof. Now the answer is obvious: altogether we have 4×4 components, 4 components on the diagonal, $3 + 2 + 1 = 6$ components above the diagonal and $3 + 2 + 1 = 6$ components under the diagonal and we know that these components are equal to components above the diagonal. Thus the final answer is $4 + 6 = 10$.

3. Proper time and physical distances

One of the most central problems in the geometry of 4-spacetime can be formulated as follows. If the metric tensor is given, how actual (measurable) time and distances are related with coordinates x^0, x^1, x^2, x^3 chosen in arbitrary way.

Proper time

Let us consider the world line of an observer who uses some clock to measure the actual or proper time $d\tau$ between two infinitesimally close events in the same place in space. Obviously we should put

$$dx^1 = dx^2 = dx^3.$$

Let us define proper time exactly as in Special Relativity:

$$d\tau = \frac{ds}{c},$$

then we have

$$ds^2 \equiv c^2 d\tau^2 = g_{ik} dx^i dx^k = g_{00} (dx^0)^2,$$

thus

$$d\tau = \frac{1}{c} \sqrt{g_{00}} dx^0.$$

For the proper time between any two events (not necessary that these events are infinitesimally close) occurring at the same point in space we have

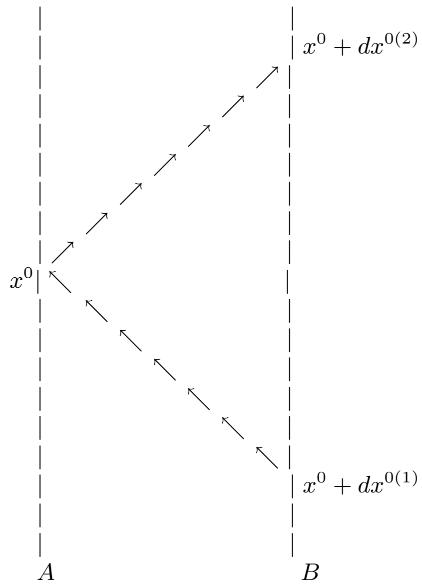
$$\tau = \frac{1}{c} \int \sqrt{g_{00}} dx^0.$$

Spatial distance

Separating the space and time coordinates in ds we have

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + 2 g_{0\alpha} dx^0 dx^\alpha + g_{00} (dx^0)^2.$$

To define dl we will use a light signal according to the following procedure: From some point B with spatial coordinates $x^\alpha + dx^\alpha$ a light signal emitted at the moment corresponding to time coordinate $x^0 + dx^{0(1)}$ propagates to a point A with spatial coordinates x^α and then after reflection at the moment corresponding to time coordinate x^0 the signal propagates back over the same path and is detected in the point B at the moment corresponding to time coordinate $x^0 + dx^{0(2)}$ as shown below.



The interval between the events which belong to the same world line of light in Special and General Relativity is always equal to zero:

$$ds = 0.$$

Solving this equation with respect to dx^0 we find two roots:

$$dx^{0(1)} = \frac{1}{g_{00}} \left(-g_{0\alpha} dx^\alpha - \sqrt{(g_{0\alpha} g_{0\beta} - g_{\alpha\beta} g_{00}) dx^\alpha dx^\beta} \right)$$

$$dx^{0(2)} = \frac{1}{g_{00}} \left(-g_{0\alpha} dx^\alpha + \sqrt{(g_{0\alpha} g_{0\beta} - g_{\alpha\beta} g_{00}) dx^\alpha dx^\beta} \right)$$

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}} \sqrt{(g_{0\alpha} g_{0\beta} - g_{\alpha\beta} g_{00}) dx^\alpha dx^\beta}.$$

Then

$$dl = \frac{c}{2} d\tau = \frac{c}{2} \frac{\sqrt{g_{00}}}{c} (dx^{0(2)} - dx^{0(1)})$$

and finally

$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta, \text{ where } \gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}}.$$

4. Covariant differentiation

In Special Relativity if A_i is a vector dA^i is also a vector (the same is valid for any tensor). But in curvilinear coordinates this is not the case:

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k$$

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k \frac{\partial^2 x'^k}{\partial x^i \partial x^l} dx^l,$$

thus dA_i is not a vector unless x'^k are linear functions of x^k (like in the case of Lorentz transformations).

Let us introduce another very useful notation:

$$, i = \frac{\partial}{\partial x^i}$$

According to the principle of covariance we can not afford to have not tensors in any physical equations, thus we should replace all differentials like

$$dA_i \text{ and } \frac{\partial A_i}{\partial x^k} \equiv A_{i,k}$$

by some corrected values which we will denote as

$$DA_i \text{ and } A_{i;k}$$

correspondingly.
In arbitrary coordinates to obtain a differential of a vector which forms a vector we should subtract vectors in the same point, not in different as we have done before. Hence we need produce a parallel transport or a parallel translation.
Under a parallel translation of a vector in galilean frame of reference its component don't change, but in curvilinear coordinates they do and we should introduce some corrections:

$$DA^i = dA^i - \delta A^i.$$

These corrections obviously should be linear with respect to all components of A_i and independently they should be linear with respect of dx^k, hence we can write these corrections as

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l,$$

where Γ^i_{kl} are called Christoffel Symbols which obviously don't form any tensor, because DA_i is the tensor while as we know dA_i is not a tensor.

Covariant Derivatives and Christoffel Symbols

In terms of the Christoffel symbols

$$DA^i = (\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k) dx^l = (A^i_{,l} + \Gamma^i_{kl} A^k) dx^l,$$

$$DA_i = (\frac{\partial A_i}{\partial x^l} - \Gamma^{il}_{\ \ k} A^k) dx^l = (A_{i,l} - \Gamma^{il}_{\ \ k} A^k) dx^l,$$

$$A^i_{;l} = \frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k = A^i_{,l} + \Gamma^i_{kl} A^k,$$

$$A_{i;l} = \frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A^k = A_{i,l} - \Gamma^k_{il} A^k.$$

To calculate the covariant derivative of tensor start with contravariant tensor which can be presented as a product of two contravariant vectors $A^i B^k$. In this case the corrections under parallel transport are

$$\delta(A^i B^k) = A^i \delta B^k + B^k \delta A^i = -A^i \Gamma^k_{lm} B^l dx^m - B^k \Gamma^i_{lm} A^l dx^n,$$

since these corrections are linear we have the same for arbitrary tensor A^{ik}:

$$\delta A^{ik} = -(A^{im} \Gamma^k_{ml} + A^{mk} \Gamma^i_{ml}) dx^l$$
$$DA^{ik} = dA^{ik} - \delta A^{ik} \equiv A^{ik}_{;\,l} dx^l,$$

hence

$$A^{ik}_{;\,l} = A^{ik}_{,l} + \Gamma^i_{ml} A^{mk} + \Gamma^k_{ml} A^{im}$$

In similar way we can obtain that

$$A^i_{k;\,l} = A^i_{k,l} - \Gamma^m_{kl} A^i_m + \Gamma^i_{ml} A^m_k, \quad \text{and} \quad A_{ik;\,l} = A_{ik,l} - \Gamma^m_{il} A_{mk} - \Gamma^m_{kl} A_{m,\,i}.$$

In the most general case when we have tensor of $m+n$ rank with m contravariant and n covariant indices the rule for calculation of the covariant derivative with respect to index p is the following

$$A^{i_1\, i_2\, \ldots\, i_m}_{j_1\, j_2\, \ldots\, j_n\, ;\, \mathbf{p}} = A^{i_1\, i_2\, \ldots\, i_m}_{j_1\, j_2\, \ldots\, j_n\, ,\, \mathbf{p}} + \Gamma^{\mathbf{i_1}}_{\mathbf{kp}} A^{\mathbf{k}\, i_2\, \ldots\, i_m}_{j_1\, j_2\, \ldots\, j_n} + \Gamma^{\mathbf{i_2}}_{\mathbf{kp}} A^{i_1\, \mathbf{k}\, \ldots\, i_m}_{j_1\, j_2\, \ldots\, j_n} + \ \ldots\ + \Gamma^{\mathbf{i}_m}_{\mathbf{kp}} A^{i_1\, i_2\, \ldots\, \mathbf{k}}_{j_1\, j_2\, \ldots\, j_n} -$$

$$-\Gamma^{\mathbf{k}}_{\mathbf{j_1}\, \mathbf{p}} A^{i_1\, i_2\, \ldots\, i_m}_{\mathbf{k}\, j_2\, \ldots\, j_n} - \Gamma^{\mathbf{k}}_{\mathbf{j_2}\, \mathbf{p}} A^{i_1\, i_2\, \ldots\, i_m}_{j_1\, \mathbf{k}\, \ldots\, j_n} - \ \ldots\ - \Gamma^{\mathbf{k}}_{\mathbf{j}_n\, \mathbf{p}} A^{i_1\, i_2\, \ldots\, i_m}_{j_1\, j_2\, \ldots\, \mathbf{k}}.$$

The Relation of the Christoffel Symbols to the Metric Tensor

So far we don't know how the Christoffel symbols depend on coordinates, however we can prove that they are symmetric in the subscripts. Let some covariant vector A_i is the gradient of a scalar ϕ, i.e. $A_i = \phi_{,i}$. Then

$$A_{k;\,i} - A_{i;\,k} = \phi_{,k,i} - \Gamma^l_{ki}\phi_{,l} - \phi_{,i,k} + \Gamma^l_{ki}\phi_{,l} = \left(\Gamma^l_{ki} - \Gamma^l_{ki}\right)\phi_{,l}.$$

In Galilean coordinates

$$\Gamma^l_{ik} = \Gamma^l_{ki} = 0,$$

hence in Galilean coordinates

$$A_{k;\,i} - A_{i;\,k} = 0,$$

but taking into account that

$$A_{k;\,i} - A_{i;\,k}$$

is a tensor we conclude that if it equals to zero in one system of coordinates it should be equal to zero in any other coordinate system, hence

$$\Gamma^l_{ik} = \Gamma^l_{ki}$$

in any coordinate system.
This is a typical example of the proof widely used in General Relativity:

If some equality between tensors is valid in one coordinate system then this equality is valid in arbitrary coordinate system. This is obvious advantage to deal with tensors.

Then we can show that covariant derivatives of g_{ik} are equal to zero. Indeed:

$$DA_i = g_{ik}DA^k$$

$$DA_i = D(g_{ik}A^k) = g_{ik}DA^k + A^k Dg_{ik},$$

hence

$$g_{ik}DA^k = g_{ik}DA^k + A^k Dg_{ik},$$

which obviously means that

$$A^k Dg_{ik} = 0.$$

Taking into account that A^k is arbitrary vector, we conclude that

$$Dg_{ik} = 0.$$

This is another example of proof in General Relativity:

If the the sum $B_{ik}A^i = 0$ for arbitrary vector A^i then the tensor $B_{ik} = 0$.

Then taking into account that

$$Dg_{ik} = g_{ik;m}dx^m = 0$$

for arbitrary infinitesimally small vector dx^m we have

$$g_{ik;m} = 0.$$

Now we are ready to relate the Christoffel symbols to the metric tensor. Introducing useful notation

$$\Gamma_{k,\,il} = g_{km}\Gamma^m_{il},$$

we have

$$g_{ik;\,l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,\,il} - \Gamma_{i,\,kl} = 0.$$

Permuting the indices i, k and l twice as

$$i \to k, \quad k \to l, \quad l \to i,$$

we have

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,\,il} + \Gamma_{i,\,kl}, \quad \frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,\,kl} + \Gamma_{l,\,ik} \ \text{ and } \ -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,\,ki} - \Gamma_{k,\,li}.$$

Taking into account that

$$\Gamma_{k,\,il} = \Gamma_{k,\,li},$$

after summation of these three equation we have

$$g_{ik,l} + g_{li,k} - g_{kl,i} = 2\Gamma_{i,\,kl},$$

and finally

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right).$$

Now we have expressions for the Christoffel symbols in terms of the metric tensor and hence we know their dependence on coordinates.

5. Physical applications

The previous material can be summarized as follows:

Gravity is equivalent to curved space-time, hence in all differentials of tensors we should take into account the change in the components of a tensor under an infinitesimal parallel transport. Corresponding corrections are expressed in terms of the Cristoffel symbols and reduced to replacement of any partial derivative by corresponding covariant derivative. In other words we can say that

If one wants to take into account all effects of Gravity on any local physical process, described by the corresponding equations written in framework of Special Relativity, one should just replace all partial derivatives by covariant derivatives in these equation according to the following very nice and simple but actually very strong and important formulae:

$$\mathbf{d} \rightarrow \mathbf{D}$$

and

$$\mathbf{,} \rightarrow \mathbf{;}$$

Example:

In special Relativity obviously

$$dg_{ik} = 0 \text{ and } g_{ik,l} = 0,$$

while in General Relativity

$$Dg_{ik} = 0 \text{ and } g_{ik;l} = 0.$$

The motion of a free particle

Let us apply above formulae to description of motion of a test particle in a given gravitational field. Let

$$u^i = \frac{dx^i}{ds}$$

is the four-velocity. Then equation for motion of a free particle in absence of gravitational field is

$$\frac{du^i}{ds} = 0$$

is generalized to the equation

$$\frac{Du^i}{ds} = 0,$$

which gives

$$\frac{Du^i}{ds} = \frac{du^i}{ds} + \Gamma^i_{kn} u^k \frac{dx^n}{ds} = \frac{d^2x^i}{ds^2} + \Gamma^i_{kn} u^k u^n = 0.$$

Thus from physical point of view the equation

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0$$

describes the motion of free particle in a given gravitational field and

$$\frac{d^2x^i}{ds^2} = -\Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds}$$

is the four-acceleration, while from geometrical point of view this equation is the equation for geodesics in a curved space-time. That is why all particles move with the same acceleration and now this experimental fact is not coincidence anymore but consequence of geometrical interpretation of gravity.