Notes 4

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Black Holes

General idea

Let us consider ds for test particle in rest, i.e. put $dr = d\theta = d\phi = 0$, in this case

$$ds^2 = g_{00} dx^{0^2},$$

If $g_{00} = 0$ then $ds^2 = 0$, which means that the world line of particle in rest is the world line of light, hence at the surface $g_{00} = 0$ no particle with finite rest mass can be in rest. Thus the surface $g_{00} = 0$ is the limit of stationarity.

Let us consider a surface $F(r) = const$ and let $n_i = F_{,i}$ is its normal.

If $g^{11} = 0$ then

$$g^{ik} n_i n_k = g^{11} n_1 n_1 = 0,$$

which means that n_i is null vector and any particle with finite rest mass can not move outward the surface $g^{11} = 0$, thus this surface is the event horizon.

Schwarzschild Black Holes

Schwarzschild metric has the following form:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right),$$

where

$$r_g = 2GM/c^2 = 3(M/M_\odot) \text{ km}, \quad \text{where } M_\odot \text{ is the mass of Sun}$$

is **the gravitational radius**.

There is no singularity at $r = r_g$.:

Using the coordinate transformations

$$c\tau = ct + \int \frac{r_g^{1/2} r^{1/2} dr}{r - r_g}, \quad R = ct + \int \frac{r^{3/2} dr}{r_g^{1/2}(r - r_g)}$$

we can show that the Schwarzschild metric is not singular at $r = r_g$ and takes the following form

$$ds^2 = c^2 d\tau^2 - \frac{r_g}{r} dR^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2).$$

Indeed, by differentiating

$$c d\tau = c dt + \frac{r_g^{1/2} r^{1/2} dr}{r - r_g}, \quad dR = c dt + \frac{r^{3/2} dr}{r_g^{1/2}(r - r_g)},$$

and then subtracting the first from the second we have

$$dR - c d\tau = \frac{dr}{r - r_g} \left(\frac{r^{3/2}}{r_g^{1/2}} - r_g^{1/2} r^{1/2} \right) =$$

$$= \frac{r^{1/2}dr}{(r-r_g)r_g^{1/2}}(r-r_g) = \left(\frac{r}{r_g}\right)^{1/2} dr,$$

hence

$$dr = \left(\frac{r_g}{r}\right)^{1/2}(dR - cd\tau).$$

Subtracting the first multiplied by r/r_g from the second we have

$$\frac{r}{r_g}cd\tau - DR = cdt(\frac{r}{r_g} - 1),$$

hence

$$cdt = \frac{crd\tau - r_g dR}{r - r_g}.$$

Then substituting the expressions for dr and cdt into ds^2 in the Schwarzschild form we obtain

$$ds^2 = \frac{r-r_g}{r}\left(\frac{rcd\tau - r_g dR}{r-r_g}\right)^2 - \frac{r_g}{r-r_g}(dR - cd\tau)^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) =$$

$$= \frac{1}{r-r_g}\left[\frac{1}{r}(rcd\tau - r_g dR)^2 - r_g(dR - cd\tau)^2\right] - r^2(d\theta^2 + \sin^2\theta d\phi^2) =$$

$$= \left[c^2d\tau^2(r-r_g) - 2cdRd\tau(\frac{r_g r}{r} - r_g) - dR^2(\frac{r_g^2}{r} - r_g)\right] - r^2(d\theta^2 + \sin^2\theta d\phi^2) =$$

$$= c^2d\tau^2 - \frac{r_g}{r}dR^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

From

$$r^{1/2}dr = r_g^{1/2}d(R - c\tau)$$

we have

$$\frac{2}{3}r^{3/2} = C + r_g^{1/2}(R - c\tau),$$

then choosing the constant of integration $C = 0$ so that $r = 0 \;\rightarrow\; R - c\tau = 0$, we have

$$r = \left[\frac{3}{2}r_g^{1/2}(R - c\tau)\right]^{2/3}.$$

Finally putting this into the metric in new coordinates we have

$$ds^2 = c^2d\tau^2 - \left[\frac{2r_g}{3(R-c\tau)}\right]^{2/3} - \left[\frac{3}{2}r_g^{1/2}(R-c\tau)\right]^{4/3}(d\theta^2 + \sin^2\theta d\phi^2),$$

we can see that the metric depends on τ, which means that the gravitational field is non- stationary.

We can see that there is no singularity at $r = r_g$.

The equation $ds^2 = 0$ for $d\theta = d\phi = 0$ gives

$$c\frac{d\tau}{dR} = \pm\frac{1}{\left[\frac{3}{2r_g}(R - c\tau)\right]^{1/3}} = \pm\sqrt{\frac{r_g}{r}}.$$

Two signs correspond to boundaries of light cone, i.e. "+" corresponds to outward photon and "-" corresponds to inward photon.

When $r > r_g$ straight line $r = \text{const}$ falls inside the light cone.

If $r < r_g$ we have $|cd\tau/dR| > 1$ so the line $r =$ const lies outside light cone, which means no particles can be at rest in this region.

We see also that all world lines intersect the line $r = 0$. Thus $r = r_g$ is event horizon and lies on light cone, hence event horizon is the null surface.

The motion of a particle in the gravitational field of a Schwarzschild black hole

The Hamilton-Jacobi equation again:

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} - m^2c^2 = 0,$$

where four-momentum $p_i = -\frac{\partial S}{\partial x^i}$ and $p_0 = E$ (energy), $p_3 = L$ (angular momentum).

It is possible to show that

$$E\left(1 - \frac{r_g}{r}\right)^{-1}\frac{dr}{dt} = c\sqrt{E^2 - \mathbf{U}^2_{\mathbf{eff}}},$$

where $\mathbf{U}_{\mathbf{eff}}$ is the "effective potential energy":

$$\mathbf{U}_{\mathbf{eff}}(r) = mc^2\sqrt{\left(1 - \frac{r_g}{r}\right)\left(1 + \frac{L^2}{m^2c^2r^2}\right)}.$$

Here L is the angular momentum and m is the mass of a particle.

$\mathbf{U}_{\mathbf{eff}}$ can be used to find stable and unstable circular orbits.
For given radius $\mathbf{U}_{\mathbf{eff}}$ is equal to the energy of a particle which has the turn point for this r, i.e. $dr/dt = 0$, thus the condition $E > \mathbf{U}_{\mathbf{eff}}$ determines the admissible range of the motion.

All circular orbits are determined by simultaneous solution of the equations

$$\mathbf{U}_{\mathbf{eff}} = E \quad \text{and} \quad \frac{d\mathbf{U}_{\mathbf{eff}}}{dr} = 0.$$

Kerr Black Holes

The Kerr metric describing the gravitational field of rotating bodies has the following form:

$$ds^2 = (1 - \frac{r_g r}{\rho^2})dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2$$

$$+\frac{2r_g r a}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the specific angular momentum.

For the Kerr metric $g_{00} = 0$ gives

$$1 - \frac{r_g r}{\rho^2} = 0,$$

thus

$$r^2 - r_g r + a^2\cos^2\theta = 0,$$

$$\Delta = r^2 - r_g r + a^2 = 0,$$

and

$$r_{st} = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2\cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2\cos^2\theta}.$$

The location of horizon in the Kerr metric: $g^{11} = 0$ ($g_{11} = \infty$) corresponds to

$$\Delta = r^2 - r_g r + a^2 = 0,$$

and

$$r = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2\cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2\cos^2\theta}.$$

$$r_{hor} = \frac{r_g}{2} \pm \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}.$$

The region between the limit of stationarity and the event horizon is called the "ergosphere". By the Penrose mechanism it is possible to extract rotational energy of Kerr black hole.

Kerr black holes and the Equivalence Principle

According to the Equivalence Principle the observer is not able to distinguish between the gravitational field and some non-inertial reference frame. In the case of the gravitational field described by the Kerr metric the corresponding non-inertial reference frame is the rotating one.

Example. How to find its angular velocity?

For $dr = 0$ and $d\theta = \pi/2$

$$ds^2 = g_{00}c^2dt^2 + 2g_{03}cdtd\phi + g_{33}d\phi^2 =$$

$$= g_{00}c^2dt^2 + g_{33}\left(d\phi^2 + \frac{2g_{03}cdtd\phi}{g_{33}} + \frac{g_{03}^2}{g_{33}^2}\right)^2 - \frac{g_{03}^2}{g_{33}}c^2dt^2 =$$

$$= \left(g_{00} - \frac{g_{03}^2}{g_{33}}\right)c^2dt^2 + g_{33}\left(d\phi + \frac{g_{03}}{g_{33}}cdt\right)^2 = \tilde{g}_{00}c^2dt^2 + g_{33}\left(d\phi - \Omega dt\right)^2,$$

where

$$\Omega = -\frac{g_{03}}{g_{33}}.$$

The following transformation of coordinates

$$\tilde{g}_{00}t = \tilde{t}, \;\; g_{33} = \tilde{r}^2, \;\; \phi - \Omega t = \tilde{\phi},$$

brings the metric to the form

$$ds^2 = c^2d\tilde{t}^2 - \tilde{r}^2d\tilde{\phi}^2,$$

which is locally galilean metric. Hence locally the observer can not discriminate between the Kerr gravitational field and non-inertial frame of reference rotating with angular velocity Ω.