Lecture 5. Black holes as solutions of the Einsten equations

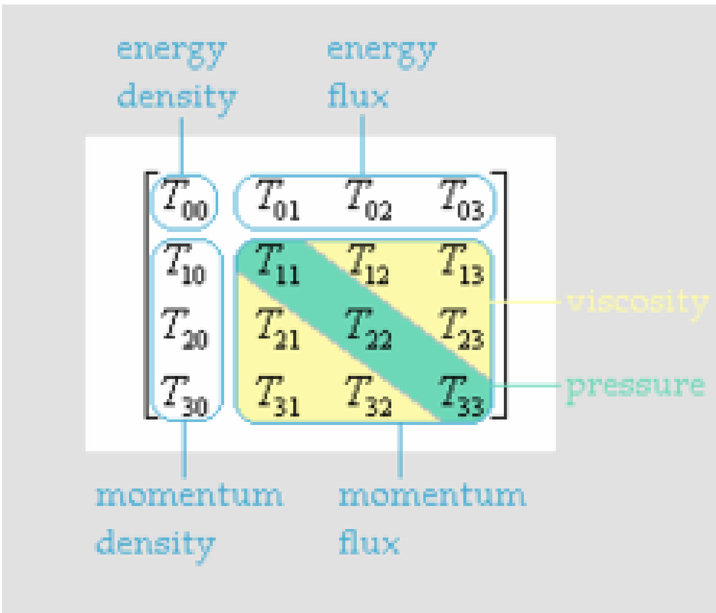
As we know from the previous lectures in curved spacetime geodesics play the same role as straight lines in flat spacetime and a freely moving particle always moves along a geodesic.

This means that in general relativity, gravity is not a force but is instead a curved spacetime geometry, where the source of curvature is the stress-energy tensor representing matter.

General relativity uses the Einstein field equations (EFE) to relate spacetime curvature to matter content in form of stress-energy tensor. These field equations are a system of partial differential equations whose solutions give all components of the metric tensor as functions of coordinates. Black holes also are solutions of the EFE.

1. Stress-Energy tensor

The stress-energy tensor (sometimes stress-energy-momentum tensor), T_{ik}, describes the density and flux of energy and momentum:



In general relativity this tensor is symmetric and contains ten independent components:

The component T^{00} represents the energy density (1 component).

The components T^{0i} (i=1,2,3) represent the flux of energy across the x^i surface, which is equivalent to T^{i0}, the density of the ith momentum (3 components).

The components T^{ij} (i,j=1,2,3) represent flux of i momentum across the x^j surface. In particular, T^{ii} represents a pressure-like quantity, normal stress (3 components), whereas T^{ij} (i is not equal to j) represents shear stress (3 components) .

All these ten components participate in generation of gravitational field, while in Newton gravity the only source of gravitational field is the mass density.

2. The Riemann curvature tensor

As we know the Christoffel symbols can be derived from the vanishing of the covariant derivative of the metric tensor and can be written in terms of the metric tenor as (see Lecture 3):

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right).$$

In the most general case when we have tensor of $m+n$ rank with m contravariant and n covariant indices the rule for calculation of the covariant derivative with respect to index p is the following

$$A^{i_1\, i_2\, \dots\, i_m}_{j_1\, j_2\, \dots\, j_n}\,;\mathbf{p} = A^{i_1\, i_2\, \dots\, i_m}_{j_1\, j_2\, \dots\, j_n}\,,\mathbf{p} + \Gamma^{\mathbf{i_1}}_{\mathbf{kp}} A^{\mathbf{k}\, i_2\, \dots\, i_m}_{j_1\, j_2\, \dots\, j_n} + \Gamma^{\mathbf{i_2}}_{\mathbf{kp}} A^{i_1\, \mathbf{k}\, \dots\, i_m}_{j_1\, j_2\, \dots\, j_n} + \dots + \Gamma^{\mathbf{i_m}}_{\mathbf{kp}} A^{i_1\, i_2\, \dots\, \mathbf{k}}_{j_1\, j_2\, \dots\, j_n} -$$

$$- \Gamma^{\mathbf{k}}_{\mathbf{j_1}\,\mathbf{p}} A^{i_1\, i_2\, \dots\, i_m}_{\mathbf{k}\, j_2\, \dots\, j_n} - \Gamma^{\mathbf{k}}_{\mathbf{j_2}\,\mathbf{p}} A^{i_1\, i_2\, \dots\, i_m}_{j_1\, \mathbf{k}\, \dots\, j_n} - \dots - \Gamma^{\mathbf{k}}_{\mathbf{j_n}\,\mathbf{p}} A^{i_1\, i_2\, \dots\, i_m}_{j_1\, j_2\, \dots\, \mathbf{k}}.$$

We know that

$$A_{i,k,l} - A_{i,l,k} = 0.$$

What can we say about the following commutator

$$A_{i;\,k;\,l} - A_{i;\,l;\,k}?$$

Straightforward calculations show (see Coursework 5) that this is not equal to zero in the presence of gravitational field and can be presented as

$$A_{i;\,k;\,l} - A_{i;\,l;\,k} = A_m R^m_{ikl},$$

where

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}.$$

You are supposed to derive all this stuff yourself and you are able to do this!

(see Coursework 5).

The object R^i_{klm} obviously is a tensor and called the curvature Riemann tensor. We know that if at least one component of a tensor is not equal to zero at least in one frame of reference, the same is true for any other frame of reference, in other words, tensors can not be eliminated by transformations of coordinates.
The Riemann tensor describes actual tidal gravitational field, which is not local and, hence, can not be eliminated even in the locally inertial frame of reference.

Example: Geodesic deviation equation

The geodesic deviation equation is an equation involving the Riemann curvature tensor, which measures the change in separation of neighbouring geodesics. In the language of mechanics it measures the rate of relative acceleration of two particles moving forward on neighbouring geodesics.

$$u^i = \frac{dx^i}{ds},$$

which has to be folded into one index. There's an infinitesimal separation vector between the two geodesics η^i, which also eats up an index on Riemann. A third (free) index is needed to exit with the rate of change of the displacement. The forth (summed) index is less obvious. It is assumed that the two geodesics are neighboring ones that start out parallel. So at first there is no relative velocity. But we have to use the velocity vector (4-velocity) along both neighbors, so it comes in twice and the last index is used:

$$\frac{d^2\eta^i}{ds^2} = R^i_{klm} u^k u^l \eta^m.$$

If gravitational field is weak and all motions are slow (i.e $u^i \approx \delta^i_0$), **the above equation is reduced to the Newtonian equation for the tidal acceleration.**

3. The Einstein field equations

It seems to be a good idea to relate the curvature Riemann tensor tensor with the stress-energy tensor? Unfortunately the rang of this tensor is 4**, which is too big in comparison with rang** 2 **for the stress-energy tensor. To solve this problem we can construct the tensor of the second rank like this:**

$$R_{ik} = g^{lm} R_{limk} = R^{l}_{ilk}$$

This tensor is called the Ricci tensor. Then we even can construct a zero rank tensor, i.e. a scalar

$$R = g^{ik} R_{ik},$$

which is called the scalar curvature.
Einstein (with help of Gilbert) introduced the following tensor

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R,$$

which is called the Einstein'tensor.
Now the Einstein Field Equations (EFE) can be written as

$$G_{ik} = \kappa T_{ik},$$

where the constant κ **is called the Einstein constant. To determine this constant we can use so called the correspondence principle, which says that the EFE reduce to Newton's law of gravity (the Poisson's equation)**

$$\Delta \phi = 4\pi G \rho$$

by using both the weak-field approximation and the slow-motion approximation. The result is

$$\kappa = \frac{8\pi G}{c^4}$$

(see coursework 5). **Finally EFE can be written as**

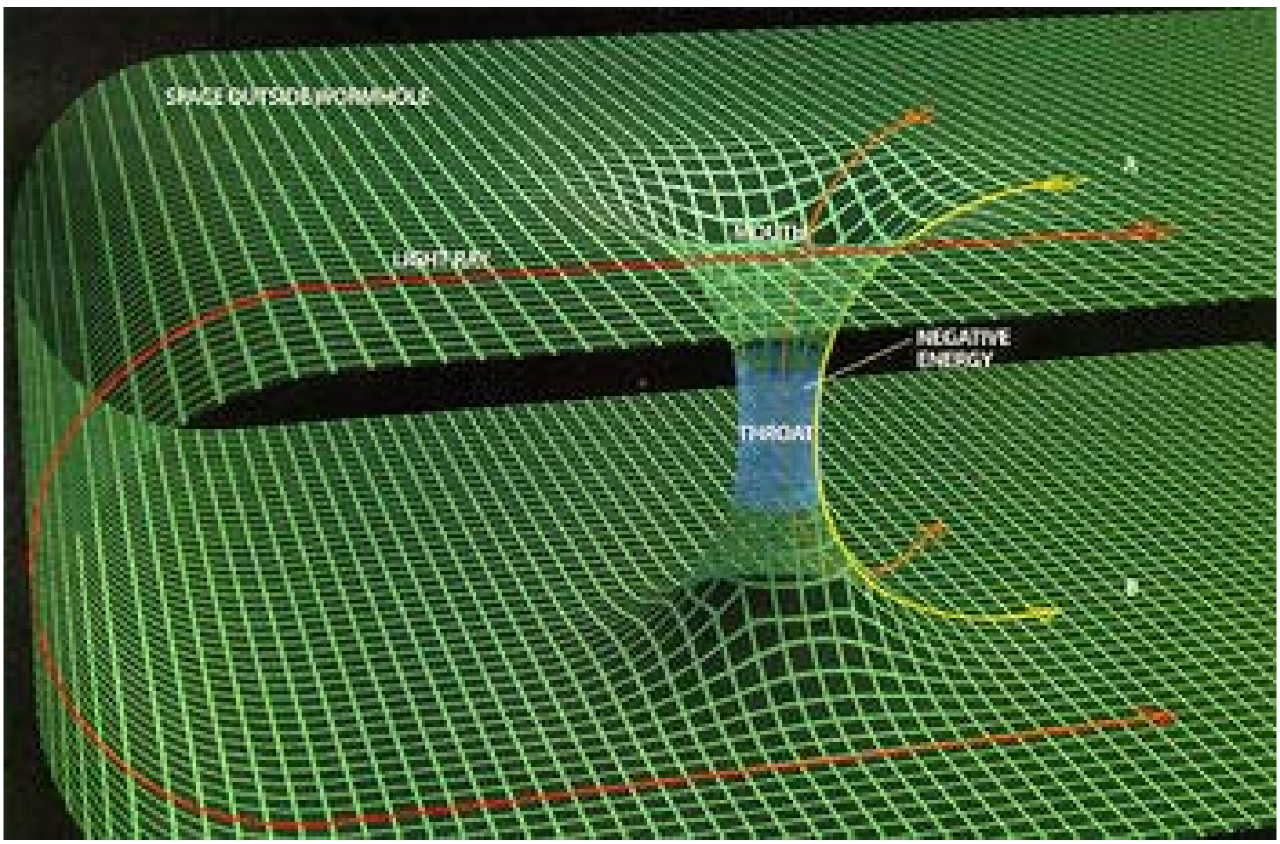
$$\mathbf{R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik}.}$$

Despite the simple appearance of the equation it is, in fact, quite complicated. Given a specified distribution of matter and energy in the form of a stress-energy tensor, the EFE are understood to be equations for the metric tensor g_{ik}**, as both the Ricci tensor and Ricci scalar depend on the metric in a complicated nonlinear manner. In fact, when fully written out, the EFE are a system of 10 coupled, nonlinear, hyperbolic-elliptic partial differential equations.**

Solutions of the Einstein field equations model extremely wide variety of gravitational fields.

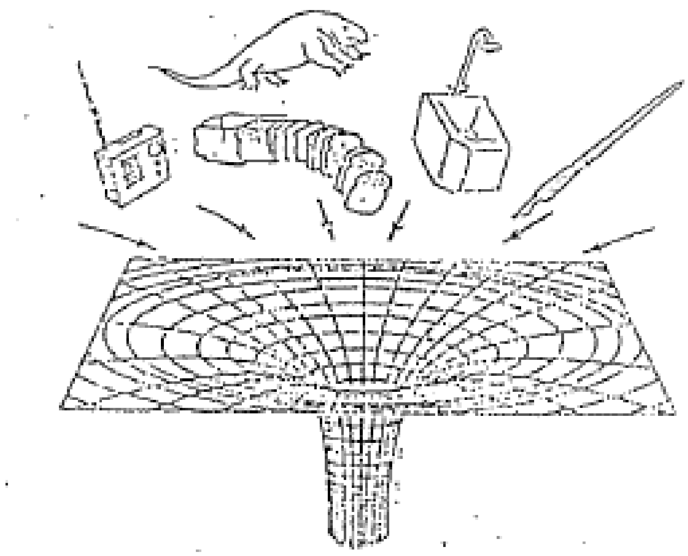
Example1. A wormhole

This is really exotic solution of EFE. In physics, a wormhole is a hypothetical topological feature of spacetime that is essentially a 'shortcut' through space and time. A wormhole has at least two mouths which are connected to a single throat. If the wormhole is traversable, matter can 'travel' from one mouth to the other by passing through the throat. While there is no observational evidence for wormholes, spacetimes containing wormholes are known to be valid solutions of the Einstein's equations.



4. No-hair theorem

The no-hair theorem says that all black hole solutions of the Einstein field equations together with Maxwell equations can be completely characterized by only three externally parameters: mass, electric charge, and angular momentum, which can not just disappear inside black holes because all these parameters are conserved quantities. All other information about the matter which formed a black hole or is falling into it, "disappears" behind the black-hole event horizon and is therefore permanently inaccessible to external observers



In other words, the gravitational field of a black hole is determined only by total mass, electric charge and angular momentum of matter from which the black hole was formed plus mass, electric charge and angular momentum of everything that fallen into black hole after its formation.

In realistic astrophysical situations electric charge of black holes is very likely equal to zero, while angular moment of black holes is very likely not equal to zero. For this reason, we will consider two types of black holes: spherically symmetric Schwarzschild (non-rotating) black hole and Kerr (rotating) black hole.

5. Schwarzschild metric

The Schwarzschild metric describes the gravitational field outside a spherical, non-rotating mass such as a non-rotating planet, star or black hole.

The Schwarzschild solution is the most general spherically symmetric, vacuum solution of the EFE for a Schwarzschild black hole, which is a black hole that has no electric charge or angular momentum.

The Schwarzschild solution is named in honor of its discoverer Karl Schwarzschild, who found the solution only about a month after the publication of Einstein's theory of general relativity and published this solution in 1916. It was the first exact solution of the EFE other than the trivial flat spacetime solution.

The Schwarzschild black hole is characterized by a surrounding spherical surface, called the event horizon, which is situated at the Schwarzschild or gravitational radius (often called the radius of a black hole).

In Schwarzschild coordinates, the Schwarzschild metric has the form:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right),$$

where

$$r_g = 2GM/c^2 = 3(M/M_\odot) \text{ km}, \quad \text{where } M_\odot \text{ is the mass of Sun}$$

is the gravitational radius.
We see that $g_{11} \to \infty$ if $r \to r_g$. The following question arises, whether this is a real singularity of spacetime at $r = r_g$ or this is just a consequence of our choice of the frame of reference. The answer is that this is coordinate rather than real singularity. Using the coordinate transformations

$$c\tau = ct + \int \frac{r_g^{1/2} r^{1/2} dr}{r - r_g}, \quad R = ct + \int \frac{r^{3/2} dr}{r_g^{1/2}(r - r_g)}$$

we can show that the metric written in new coordinates is not singular at $r = r_g$ and takes the following form

$$ds^2 = c^2 d\tau^2 - \frac{r_g}{r} dR^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Indeed, by differentiating

$$cd\tau = cdt + \frac{r_g^{1/2} r^{1/2} dr}{r - r_g}, \quad dR = cdt + \frac{r^{3/2} dr}{r_g^{1/2}(r - r_g)},$$

and then subtracting the first from the second we have

$$dR - cd\tau = \frac{dr}{r - r_g}\left(\frac{r^{3/2}}{r_g^{1/2}} - r_g^{1/2} r^{1/2}\right) =$$

$$= \frac{r^{1/2} dr}{(r - r_g) r_g^{1/2}} (r - r_g) = \left(\frac{r}{r_g}\right)^{1/2} dr,$$

hence

$$dr = \left(\frac{r_g}{r}\right)^{1/2} (dR - cd\tau).$$

Subtracting the first multiplied by r/r_g from the second we have

$$\frac{r}{r_g} cd\tau - DR = cdt\left(\frac{r}{r_g} - 1\right),$$

hence

$$cdt = \frac{crd\tau - r_g dR}{r - r_g}.$$

Then substituting the expressions for dr and cdt into ds^2 in the Schwarzschild form we obtain

$$ds^2 = \frac{r - r_g}{r}\left(\frac{rcd\tau - r_g dR}{r - r_g}\right)^2 - \frac{r_g}{r - r_g}(dR - cd\tau)^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) =$$

$$= \frac{1}{r - r_g}\left[\frac{1}{r}(rcd\tau - r_g dR)^2 - r_g(dR - cd\tau)^2\right] - r^2(d\theta^2 + \sin^2\theta d\phi^2) =$$

$$= \left[c^2 d\tau^2(r - r_g) - 2cdRd\tau(\frac{r_g r}{r} - r_g) - dR^2(\frac{r_g^2}{r} - r_g)\right] - r^2(d\theta^2 + \sin^2\theta d\phi^2) =$$

$$= c^2 d\tau^2 - \frac{r_g}{r}dR^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

From

$$r^{1/2}dr = r_g^{1/2}d(R - c\tau)$$

we have

$$\frac{2}{3}r^{3/2} = C + r_g^{1/2}(R - c\tau),$$

then choosing the constant of integration $C = 0$ so that $r = 0$ corresponds to $R - c\tau = 0$, we have

$$r = \left[\frac{3}{2}r_g^{1/2}(R - c\tau)\right]^{2/3}.$$

Finally, putting this into the metric written in terms of new coordinates, we have

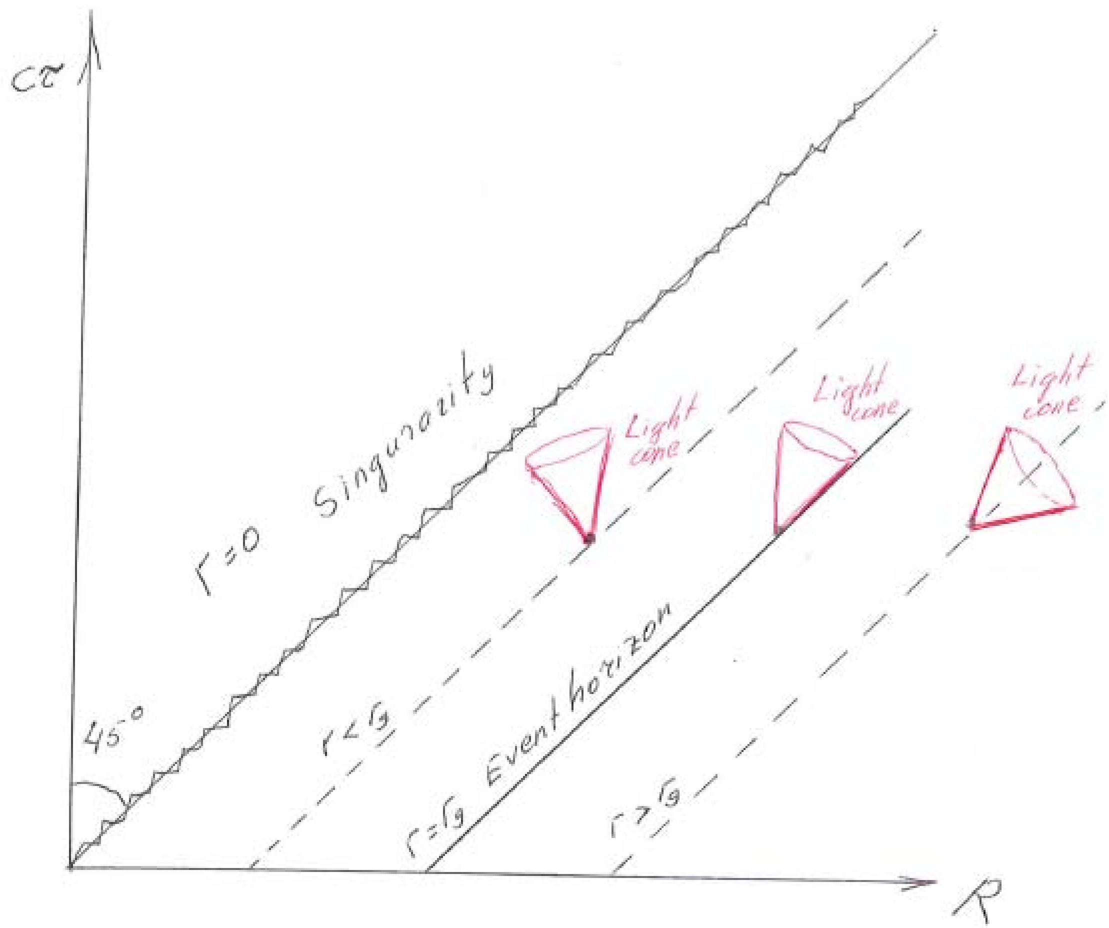
$$ds^2 = c^2 d\tau^2 - \left[\frac{2r_g}{3(R - c\tau)}\right]^{2/3} - \left[\frac{3}{2}r_g^{1/2}(R - c\tau)\right]^{4/3}(d\theta^2 + \sin^2\theta d\phi^2),$$

we can see that the metric in these new coordinates depends on the time coordinate, τ, which means that the gravitational field is non- stationary in these new coordinates. Thus we can conclude that there is no physical singularity at $r = r_g$.
Let us now consider the structure of light cones in the in these new coordinates. The equation $ds^2 = 0$ for $d\theta = d\phi = 0$ gives

$$c\frac{d\tau}{dR} = \pm\frac{1}{\left[\frac{3}{2r_g}(R - c\tau)\right]^{1/3}} = \pm\sqrt{\frac{r_g}{r}}.$$

Two signs correspond to boundaries of light cone, i.e. "+" corresponds to outward photon and "-" corresponds to inward photon.
When $r > r_g$ straight line $r = \text{const}$ falls inside the light cone.
If $r < r_g$ we have $|cd\tau/dR| > 1$ so the line $r = \text{const}$ lies outside light cone, which means no particle can be at rest in this region.

We see also that all world lines intersect the line r=0, which is the real physical singularity, while the line $r=r_g$, the event horizon, lies on the light cone, but does not contain any singularity. We see that event horizon corresponds to world line of light, hence the event horizon is the null surface:



6. Kerr Black Holes

The Kerr metric describing the gravitational field of a rotating black hole has the following form:

$$ds^2 = (1 - \frac{r_g r}{\rho^2})dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2$$

$$+\frac{2r_g r a}{\rho^2}\sin^2\theta d\phi dt,$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta,$$

$$\Delta = r^2 - r_g r + a^2,$$

and

$$a = \frac{J}{mc},$$

where J is the specific angular momentum of the Kerr black hole
For the Kerr metric the limit of stationarity can be determined from

$$g_{00} = 1 - \frac{r_g r}{\rho^2} = 0,$$

thus

$$r^2 - r_g r + a^2\cos^2\theta = 0,$$

and finally, taking the largest solution (outer limit of stationarity)

$$r_{st} = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2\cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2\cos^2\theta}.$$

The location of the event horizon in the Kerr metric, corresponding to

$$g^{11} = 0,$$

which means that

$$g_{11} = \infty,$$

is determined from

$$\Delta = r^2 - r_g r + a^2 = 0,$$

hence

$$r = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2 \cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2 \cos^2\theta},$$

finally, taking the largest solution (outer horizon)

$$r_{hor} = \frac{r_g}{2} \pm \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}.$$

One can see that the event horizon is a spherically symmetric surface, since there is no dependence on the angle coordinate, while the limit of stationarity is oblate cylindrically symmetric surface. There are also the inner horizon and the inner limit of stationarity, both are inside the outer event horizon and hence they are not interesting for Relativistic Astrophysics.
One can see easily that

$$r_{st} \geq r_{hor}.$$

For example,

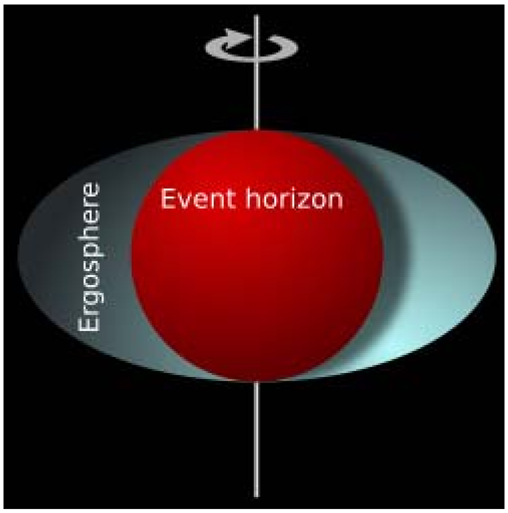
$$r_{st} = r_{hor}, \text{ if } \theta = 0, \text{ or } \theta = \pi \text{ (at the poles)},$$

and

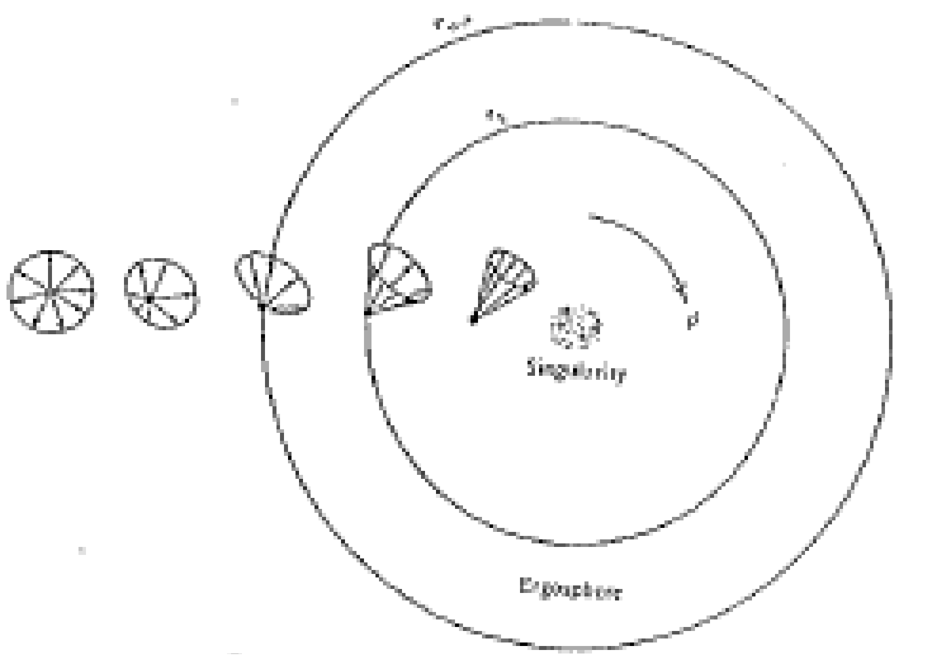
$$r_{st} = 2r_g > r_{hor}, \text{ if } \theta = \frac{\pi}{2} \text{ (at the equator)}.$$

7. Ergosphere and the Penrose Process

The region between the limit of stationarity and the event horizon is called the "ergosphere".

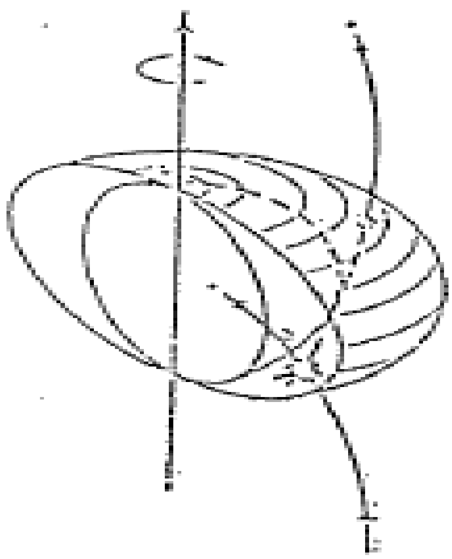


The structure of light cones if to look from the pole direction looks like this



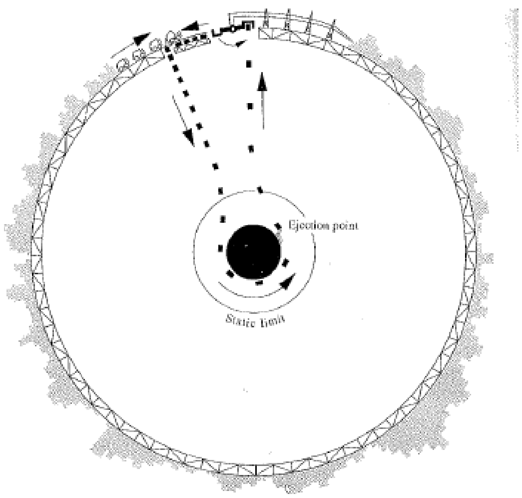
7. Penrose process

The Penrose process is a process considered by Roger Penrose wherein energy can be extracted from a rotating black hole. That extraction is made possible because the rotational energy of the black hole is located not inside the event horizon, but outside in a curl gravitational field. Such field is also called "gravimagnetic'' field. All objects in the ergosphere are unavoidably dragged by the rotating spacetime. Imagine that some body enters into the black hole and then it is split there into two pieces. The momentum of the two pieces of matter can be arranged so that one piece escapes to infinity, whilst the other falls past the outer event horizon into the black hole. The escaping piece of matter can have greater mass-energy than the original infalling piece of matter or. In other words, the captured piece has negative mass-energy.



The Penrose process results in a decrease in the angular momentum of the black hole, and that reduction corresponds to a transference of energy whereby the momentum lost is converted to energy extracted.
In result of the Penrose process the black hole can eventually lose all of its angular momentum, becoming non-rotating black hole.

Example. **May be in some remote future the Penrose process can be used in practice. Imagine that an advanced civilization found a rotating black hole …**



An advanced civilization has constructed a rigid framework around a black hole, and has built a huge city on that framework. Each day trucks carry one million tons of garbage out of the city to the garbage dump. At the dump the garbage is shoveled into shuttle vehicles which are then, one after another, dropped toward the center of the black hole. Dragging of inertial frames whips each shuttle vehicle into a circling, inward-spiraling orbit near the horizon. When it reaches a certain "ejection point," the vehicle ejects its load of garbage into an orbit of negative energy-at-infinity, $E_{\text{garbage}} < 0$. As the garbage flies down the hole, changing the hole's total mass-energy by $\Delta M = E_{\text{garbage ejected}} < 0$, the shuttle vehicle recoils from the ejection and goes flying back out with more energy-at-infinity than it took down

$$E_{\text{vehicle out}} = E_{\text{vehicle + garbage down}} - E_{\text{garbage ejected}}$$
$$> E_{\text{vehicle + garbage down}}.$$

The vehicle deposits its huge kinetic energy in a giant flywheel adjacent to the garbage dump; and the flywheel turns a generator, producing electricity for the city, while the shuttle vehicle goes back for another load of garbage. The total electrical energy generated with each round trip of the shuttle vehicle is

$$\begin{aligned}(\text{Energy per trip}) &= E_{\text{vehicle out}} - (\text{rest mass of vehicle})\\ &= (E_{\text{vehicle + garbage down}} - E_{\text{garbage ejected}}) - (\text{rest mass of vehicle})\\ &= (\text{rest mass of vehicle} + \text{rest mass of garbage} - \Delta M) - (\text{rest mass of vehicle})\\ &= (\text{rest mass of garbage}) + (\text{amount, } -\Delta M\text{, by which hole's mass decreases}).\end{aligned}$$

Thus, not only can the inhabitants of the city use the black hole to convert the entire rest mass of their garbage into kinetic energy of the vehicle, and thence into electrical power, but they can also convert some of the mass of the black hole into electrical power!