LECTURE 6. PART I

EXPERIMENTS IN GENERAL RELATIVITY

General relativity is currently the most successful gravitational theory, being almost universally accepted and well-supported by observations. General relativity's first success was in explaining the anomalous perihelion precession of Mercury, then observations of stars near the eclipsed Sun quantitatively confirmed general relativity's prediction that massive objects bend light. Other observations and experiments have since confirmed many of the predictions of general relativity, including the gravitational redshift of light and gravitational time dilation.

All these effects in the Solar System were then observed in tremendously magnified version in binary pulsars. All these issues will be considered in the Part 1.

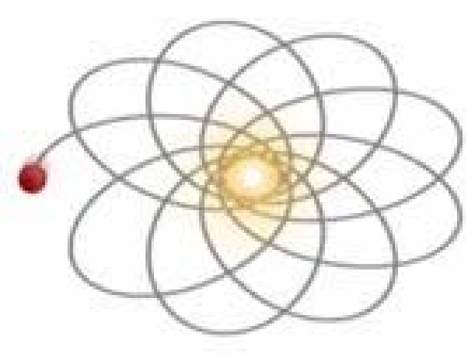
1. Classical tests

In 1916 Einstein proposed three famous tests of general relativity, subsequently called the classical tests of general relativity:

1. **the perihelion precession of Mercury's orbit**
2. **the deflection of light by the Sun**
3. **the gravitational redshift of light**

1.1 Perihelion precession of Mercury

In Newtonian physics, a lone object orbiting a spherical mass would trace out an ellipse with the spherical mass at a focus. The point of closest approach, called the perihelion in the solar system, is fixed. There are a number of solar system effects that cause the perihelion of a planet to precess, or rotate around the sun. These are mainly because of the presence of other planets, which perturb orbits. Another effect is solar oblateness, which produces only a minor contribution. The precession of the perihelion of Mercury was a longstanding problem in celestial mechanics. Careful observations of Mercury showed that the actual value of the precession disagreed with that calculated from Newton's theory by 43 seconds of arc per century, which was much larger than the experimental error at the time. In general relativity, this orbit will precess, or change orientation within its plane, due to the curvature of spacetime. This change is described as a precession of the perihelion:

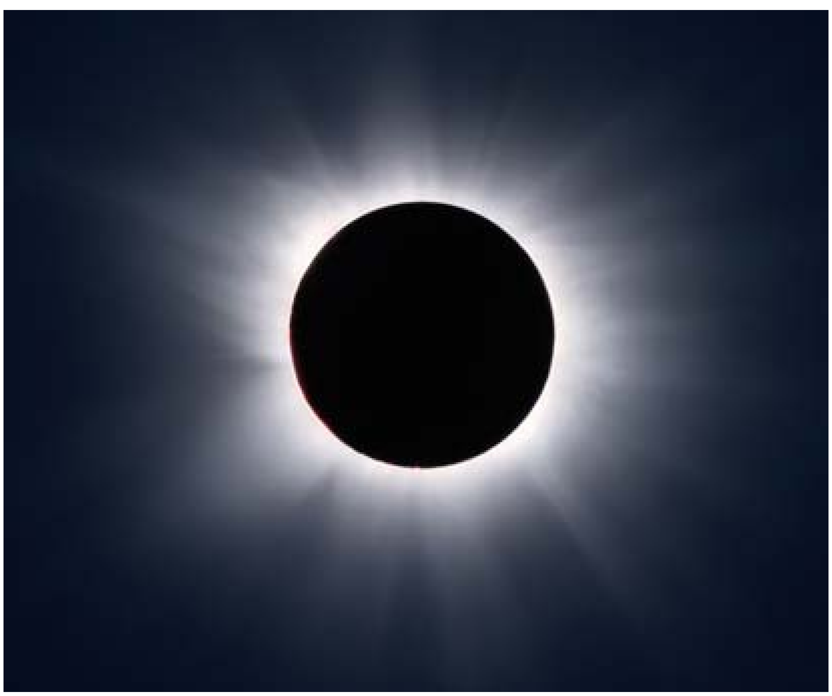


(very exaggerated)

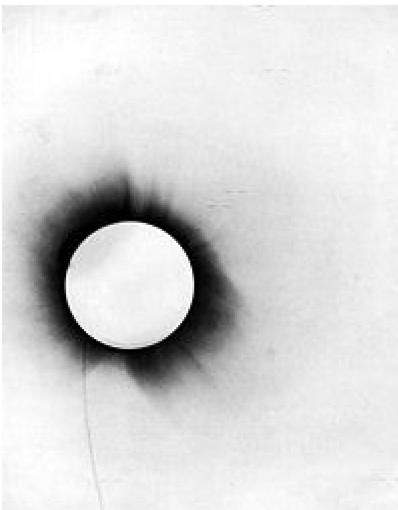
1.2. Deflection of light by the Sun

The first observation of light deflection was performed by noting the change in position of stars as they passed near the Sun on the celestial sphere. The observations were performed by Sir Arthur Eddington.

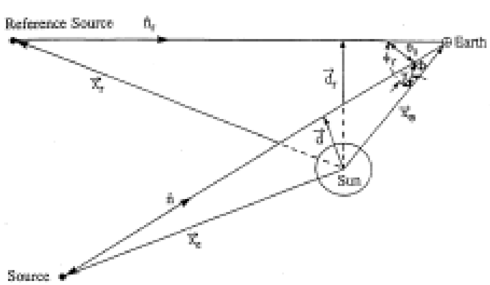
Eddington travelled to the island of Príncipe near Africa to watch the solar eclipse of May 29, 1919.



During the eclipse, he took pictures of the stars in the region around the Sun. According to the theory of general relativity, stars near the Sun would appear to have been slightly shifted because their light had been curved by its gravitational field.



Eddington's 1919 measurements of the bending of star-light by the Sun's gravity led to the acceptance of general relativity worldwide.



This effect is noticeable only during an eclipse, since otherwise the Sun's brightness obscures the stars. Newtonian gravitation predicted half the shift of general relativity.

1.3. Gravitational redshift of light

Einstein predicted the gravitational redshift of light in 1907. The more often used exact equation for gravitational redshift applies to the case outside of a non-rotating, uncharged mass which is spherically symmetric. The equation is:

$$z = \frac{1}{\sqrt{1 - \left(\frac{2GM}{rc^2}\right)}} - 1$$

where *G* is the gravitational constant, *M* is the mass of the object creating the gravitational field, *r* is the radial coordinate of the observer (which is analogous to the classical distance from the center of the object, but is actually a Schwarzschild coordinate), and *c* is the speed of light.

It was conclusively tested when the Pound-Rebka experiment in 1959 measured the relative redshift of two sources situated at the top and bottom of Harvard University's Jefferson tower.

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Jefferson laboratory at Harvard University. The experiment occurred in the left "tower".

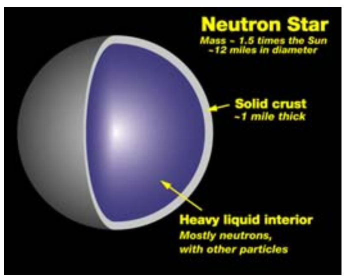
The result was in excellent agreement with general relativity. This was one of the first precision experiments testing general relativity.

2. Binary Pulsar

General relativity has been extremely well tested after 1974, when Hulse and Taylor discovered the first binary pulsar.

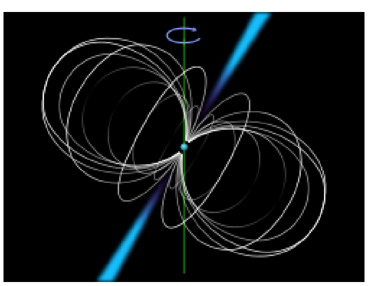
Pulsars are highly magnetized rotating neutron stars.

A neutron star is formed from the collapsed remnant of a massive star and consists mostly of neutrons. It is a very hot star supported by the Pauli exclusion principle repulsion between neutrons.



A typical neutron star has a mass between 1.35 and about 2.1 solar masses, with a corresponding radius between 10 and 20 km — 30,000 to 70,000 times smaller than the Sun. Thus, neutron stars have overall densities of 10^{17} to 10^{18} kg/m³, which compares with the approximate density of an atomic nucleus.

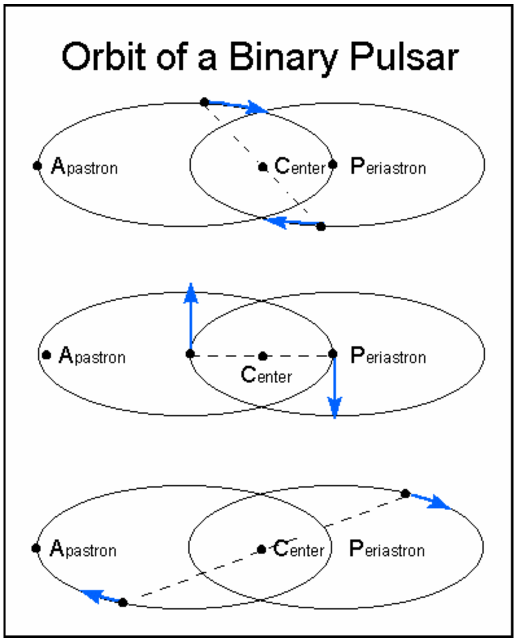
Pulsars emit a beam of detectable electromagnetic radiation in the form of radio waves. Their observed periods range from 1 ms to 10 s.



Schematic view of a pulsar. The sphere in the middle represents the neutron star, the curves indicate the magnetic field lines and the protruding cones represent the emission beams.

The radiation can only be observed when the beam of emission is pointing towards the Earth. This is called the lighthouse effect and gives rise to the pulsed nature that gives pulsars their name. Because neutron stars are very dense objects, the rotation period and thus the interval between observed pulses are very regular. For some pulsars, the regularity of pulsation is as precise as an atomic clock.

A binary pulsar is a pulsar with a binary companion, often another pulsar, white dwarf or neutron star.

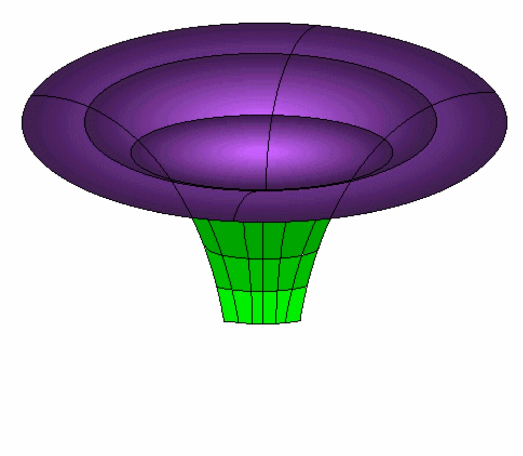


The binary pulsars allow astrophysicists to test general relativity in the case of a strong gravitational field. The timing of the pulses from the pulsar can be measured with extraordinary accuracy. A relatively simple 10-parameter model incorporating information about the pulsar timing, the Keplerian orbits and three post-Keplerian corrections (the rate of periastron advance, a factor for gravitational redshift and a rate of change of the orbital period due to gravitational radiation) is sufficient to completely model the pulsar timing.

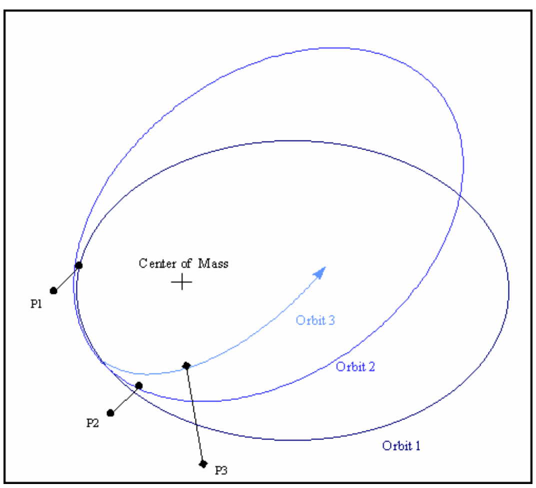
Binary pulsar timing has thus indirectly confirmed the existence of gravitational radiation and verified Einstein's general theory of relativity in a previously unknown regime.

The first binary pulsar, PSR 1913+16 or the "Hulse-Taylor binary pulsar" was discovered in 1974 at Arecibo by Joseph Hooton Taylor, Jr. and Russell Hulse, for which they won the 1993 Nobel Prize in Physics. Pulses from this system have been tracked, without glitches, to within 15 μs since its discovery.

Space-time in the vicinity of the pulsar is greatly warped



This curvature causes the pulsar orbit to advance.



The rotation of the pulsar's periastron is analogous to the advance of the perihelion of Mercury in its orbit. The observed advance for PSR 1913+16 is about 4.2 degrees per year; the pulsar's periastron advances in a single day by the same amount as Mercury's perihelion advances in a century.

Example 1. Gravity Probe A

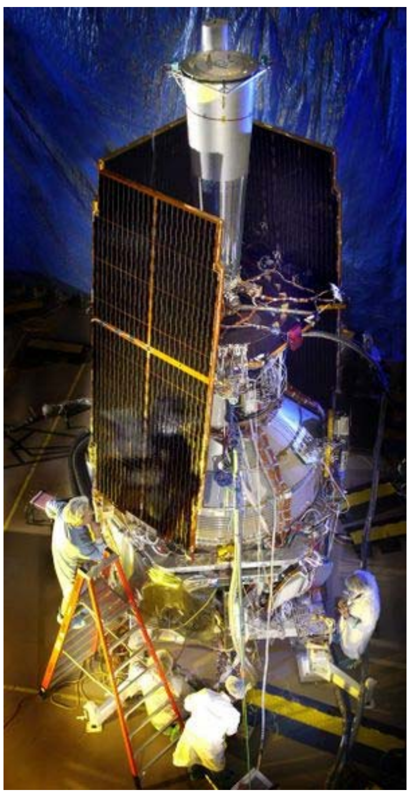
There were a lot of other precision tests of general relativity, not discussed here. I will give you just two other important examples are the Gravity Probe A satellite, launched in 1976,

which showed that gravity and velocity affect the ability to synchronize the rates of clocks orbiting a central mass.

Example 2. Gravity Probe B

General relativity predicts that rotating bodies drag spacetime around themselves in a phenomenon referred to as frame-dragging (or gravimagnetism): the rotation of an object would alter space and time, dragging a nearby object out of position compared to the predictions of Newtonian physics. The predicted effect is incredibly small — about one part in a few trillion. In order to detect it, it is necessary to look at a very massive object, like rotating black holes (see the previous lecture) or build an instrument that is incredibly sensitive.

The Gravity Probe B satellite, launched in 2004



is currently attempting to detect frame dragging. The experiment planned to check, very precisely, tiny changes in the direction of spin of four gyroscopes contained in an Earth satellite orbiting at 650 km altitude and crossing directly over the poles. They were intended to measure how space and time are "warped" by the presence of the Earth, and, more profoundly, if and how much the Earth's rotation "drags" space-time around with it.

Lecture 6. Part II. Kepler problem in general relativity

The Kepler problem in general relativity involves solving for the motion of two gravitationally interacting spherical bodies.. In this lecture we will consider the curved space-time described by the Schwarzschild metric (see the previous lecture), however the results could be generalized for the Kerr metric as well.

1. The Kepler problem in the Newtonian theory

Let start with astrophysical systems when we can use the Newtonian theory of gravity, despite that one of the companions could be a black hole. We will consider two examples. In the first example we have binary system with one invisible component, which could be a black hole. In another example an invisible object, which is suspected to be a black hole is surrounded by not only one but many visible stars. In both examples the distances between al objects in corresponding systems are considerably larger than the gravitational radius of the black hole and in the zero approximation the Newtonian theory works good enough. [We don't need to assume that some masses are small in comparison with the mass of invisible object .]

Example 1. A binary system with an invisible compact object

Let us consider a binary system in which one component is some well observed star, say, red giant, while another component is absolutely invisible object, which could be a white dwarf, neutron star or black hole. Let M_x is the mass of the invisible compact object, M is the mass of a visible star, T is the period of the orbit, i is the angle between the normal to the plane of the orbit and the line of sight to the observer and v is the projection of the orbital velocity of the visible star on the line of sight.

In observations of binaries one can measure directly only v (using Doppler shift of spectral lines) and T (using clocks).

The objective is to determine M_x.

Both the components move around the centre of mass of the binary system. Let assume for simplicity that both components move along circular orbits of radii r_x and r correspondingly.

Then, from the Newtonian theory we have

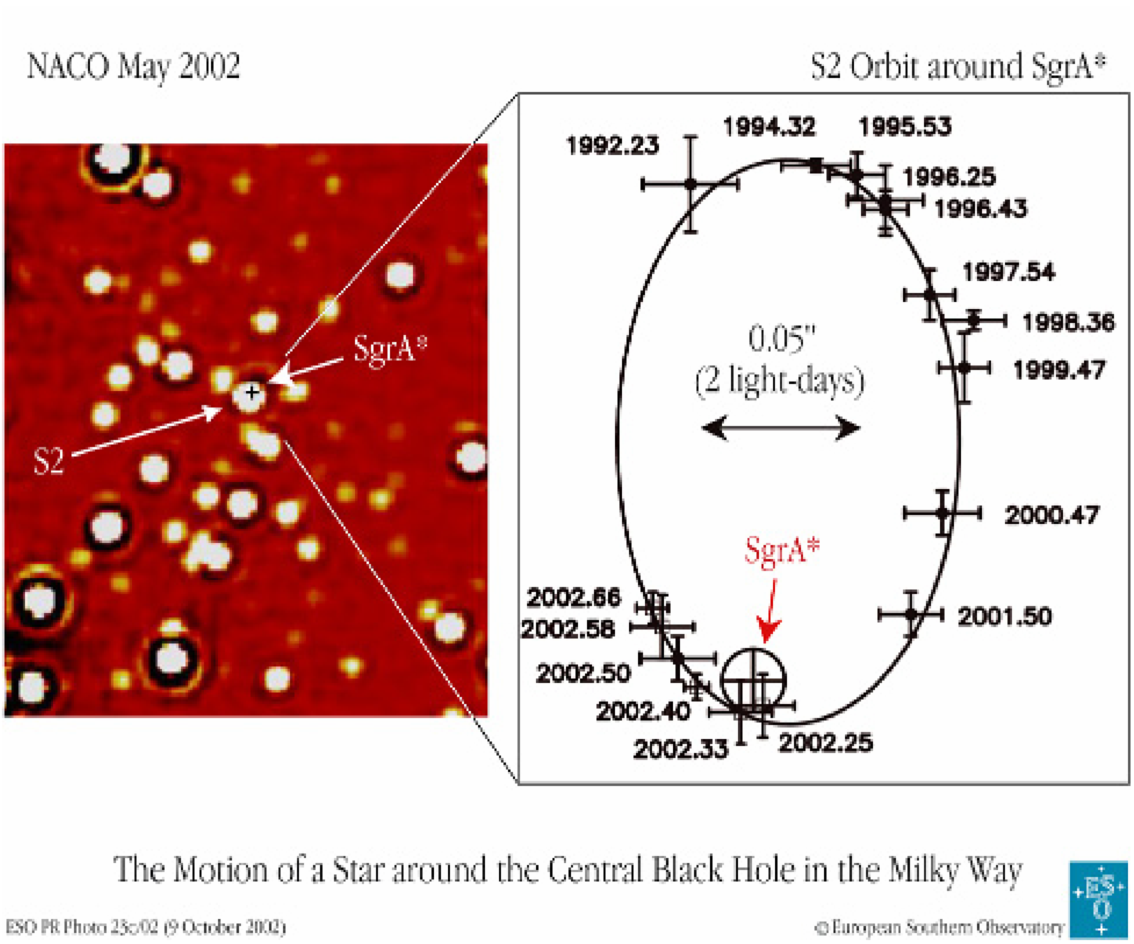
$$\begin{gathered} \mathrm{rM} = \mathrm{r_X M_X}, \\ \omega^2 \mathrm{r_X} = \mathrm{GM}(\mathrm{r_X} + \mathrm{r})^{-2}, \\ \omega^2 \mathrm{r} = \mathrm{GM_X}(\mathrm{r_X} + \mathrm{r})^{-2}, \\ \mathrm{v} = \omega \mathrm{r} \sin \mathbf{i}, \\ \omega = \frac{2\pi}{\mathrm{T}}. \end{gathered}$$

Since r_x and r can not to be determined directly, let us discriminate them from these equations to obtain so called the mass function:

$$\mathrm{f} \equiv \frac{(\mathrm{M_X} \sin \mathbf{i})^3}{(\mathrm{M_X} + \mathrm{M})^2} = \frac{\mathrm{v}^3 \mathrm{T}}{2\pi \mathrm{G}}.$$

Thus, from observations of newtonian binaries we can not determine M_x, M and i separately. Observations of binaries give only the combination of these three quantities, the mass function. We will see below that due to the relativistic effects neglected in this example, one does can determine M_x, M and i separately.

Example 2. Observable motion of star around Black Hole in the Milky Way



The Motion of a Star around the Central Black Hole in the Milky Way

2. Hamilton–Jacobi equation

As mentioned above, the Kepler problem in general relativity involves solving for the motion of two gravitationally interacting spherical bodies. One body (like a planet) is assumed to have a mass **m** that is negligible compared to the mass **M** of the other body (like the Sun). The role of the lighter object can be played by a planet around a star or a star around a massive black hole, or by photon by a neutron star or black hole. The heavier body contributes to the curvature of space-time. We know that the motion of the lighter body ("the particle" or photon) is described by the space-time geodesics of the Schwarzschild metric.

These geodesic solutions account for the anomalous precession of the planet Mercury, describe the deflection of light in a gravitational field and so on.

The geodesic equations are very useful for physical understanding of motion of particles and propagation of photons in the gravitational field. However, it is easier to work with the Hamilton–Jacobi equation. The advantage of this approach is that it equates the motion of the particle with the propagation of a wave.

The derivation of Hamilton-Jacobi equation is really very simple:

From the definition of the four-velocity

$$u^i = \frac{dx^i}{ds}$$

we have

$$ds^2 = g_{ik}dx^i dx^k = g_{ik}u^i u^k ds^2 = u_i u^i ds^2,$$

hence

$$u^i u_i = 1.$$

Four-momentum of the particle is defined as

$$p^i = mcu^i,$$

hence

$$p_i p^i = g^{ik} p_i p_k = m^2 c^2.$$

Taking into account that a covariant vector transforms as the gradient of a scalar, we can introduce such scalar function that

$$p_i = -\frac{\partial S}{\partial x^i}.$$

Then we immediately obtain the Hamilton- Jacobi Equation for a particle in a gravitational field

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} - mc^2 = 0$$

The definition of 4-velocity is not applicable to the propagation of light since

$$ds = 0.$$

We can introduce some scalar parameter λ **varying along world line of the light signal and introduce then a vector**

$$k^i = \frac{dx^i}{d\lambda},$$

which is tangent to the word line. This vector is called four- dimensional wave vector. In absence of gravitational field according to the geometrical optics the propagation of light is given by equation

$$dk^i = 0.$$

We know that the generalization of this equation in General Relativity is straightforward:

$$d \to D$$

From

$$Dk^i = 0$$

we obtain

$$\frac{dk^i}{d\lambda} + \Gamma^i_{kl} k^k k^l = 0.$$

From the definition of the four-vector for light we have

$$ds^2 = g_{ik}dx^i dx^k = g_{ik}k^i k^k d\lambda^2,$$

then taking into account that

$$ds = 0$$

Substituting covariant vector

$$k_i = -\frac{\partial \psi}{\partial x^i},$$

we obtain the Eikonal Equation in gravitational field

$$g^{ik}\frac{\partial \Psi}{\partial x^i}\frac{\partial \Psi}{\partial x^k} = 0.$$

The physical meaning of (called the Eikonal) follows from

$$\psi = -\int k_i dx^i,$$

this looks like the phase of electromagnetic wave. We can see that the General Relativity can easily solve the problem of propagation of electromagnetic signals in presence of gravitational field, while the Newtonian gravity can not even offer more or less self consistent approach to the problem. The sh ortest way to obtain the equation for propagation of light is just to put m=0 in the Hamilton–Jacobi equation and change notations.

3. The motion of a particle in the Schwarzschild metric

Taking into account spherical symmetry of the Schwarzschild metric we can choose our spherical coordinates in a such way that the plane of orbit coincides with the equatorial plane $\theta = \pi/2$. Then the Hamilton-Jacobi equation in the Schwarzschild metric can be written as

$$\left(1-\frac{r_g}{r}\right)^{-1}\left(\frac{\partial S}{c\partial t}\right)^2-\left(1-\frac{r_g}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2-\frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2-m^2c^2=0.$$

Since all coefficients in this equation do not depend on t and ϕ **we can say that**

$$\frac{\partial S}{\partial t}=-E, \text{ and } \frac{\partial S}{\partial \phi}=L,$$

where E **and** L **are constants, which by definition are the energy and angular momentum of the particle under consideration. Then putting**

$$S=-Et+L\phi+S_r(r)$$

into the Hamilton-Jacobi equation we have

$$\left(1-\frac{r_g}{r}\right)^{-1}\frac{E^2}{c^2}-\left(1-\frac{r_g}{r}\right)\left(\frac{dS_r(r)}{dr}\right)^2-\frac{L^2}{r^2}-m^2c^2=0,$$

hence

$$\frac{dS_r(r)}{dr}=\left(1-\frac{r_g}{r}\right)^{-1/2}\sqrt{\left(1-\frac{r_g}{r}\right)^{-1}\frac{E^2}{c^2}-\frac{L^2}{r^2}-m^2c^2}=$$

$$=\left(1-\frac{r_g}{r}\right)^{-1}\sqrt{\frac{E^2}{c^2}-\left(1-\frac{r_g}{r}\right)\left(\frac{L^2}{r^2}+m^2c^2\right)}.$$

Then the contravariant components of the four-momentum are

$$p^0 \equiv mc\frac{dx^0}{ds} = mc\frac{cdt}{ds} = g^{00}p_0 = \left(1-\frac{r_g}{r}\right)^{-1}\frac{\partial S}{c\partial t} = -\frac{E}{c}\left(1-\frac{r_g}{r}\right)^{-1},$$

$$p^1 \equiv mc\frac{dx^1}{ds} = mc\frac{dr}{ds} = g^{11}p_1 = -mc\left(1-\frac{r_g}{r}\right)\frac{\partial S}{\partial r} =$$

$$= -mc\left(1-\frac{r_g}{r}\right)^{1/2}\sqrt{\left(1-\frac{r_g}{r}\right)^{-1}\frac{E^2}{c^2} - \frac{L^2}{r^2} - m^2c^2} =$$

$$-mc\sqrt{\frac{E^2}{c^2} - \left(1-\frac{r_g}{r}\right)\left(\frac{L^2}{r^2} + m^2c^2\right)},$$

$$p^3 \equiv mc\frac{dx^3}{ds} = mc\frac{d\phi}{ds} = g^{33}p_3 = -\frac{1}{r^2}\frac{\partial S}{\partial \phi} = -\frac{L}{r^2}.$$

Then we can rewrite above equations as

$$\frac{dt}{ds} = -\frac{E}{mc^3}\left(1-\frac{r_g}{r}\right)^{-1},$$

$$\frac{dr}{ds} = -\frac{1}{c}\sqrt{E^2 - U_{eff}^2},$$

$$\frac{d\phi}{ds} = -\frac{L}{mcr^2},$$

where

$$U_{eff} = mc^2\sqrt{\left(1+\frac{L^2}{m^2c^2r^2}\right)\left(1-\frac{r_g}{r}\right)}$$

is called the "effective potential energy". For given radius U_{eff} is equal to the energy of a particle which has the turn point ($\frac{dr}{d\phi} = 0$), i.e. Apastron or Periastron, for this r. Indeed

$$\frac{dr}{d\phi} = \frac{mc}{Lr^2}\sqrt{E^2 - U_{eff}^2},$$

hence, if

$$\frac{dr}{d\phi} = 0, \quad \text{then} \ \ U_{eff} = E.$$

Thus the condition

$$E > U_{eff}$$

determines the admissible range of the motion. The effective potential includes in relativistic manner potential energy plus kinetic energy of non-radial motion, this kinetic energy is determined by angular momentum L.

Example. Stable and unstable circular orbits.

The radius of the stable circular orbit is obtained from the simultaneous solution of the equations

$$U_{eff} = E \ \ and \ \ \frac{dU_{eff}}{dr} = 0.$$

From

$$dU_{eff}/dr = 0$$

we have

$$dU_{eff}^2/du = 0,$$

where $u = 1/r$. **Hence**

$$-r_g\left(1+\frac{L^2u^2}{m^2c^2}\right)+(1-r_gu)\,\frac{2L^2u}{m^2c^2}=0, \ \ or \ \ r_gr^2+3r_g\left(\frac{L}{mc}\right)^2-2\left(\frac{L}{mc}\right)^2r=0.$$

Solving this equation we have

$$r_{\pm}=\frac{L^2}{m^2c^2r_g}\pm\sqrt{\left(\frac{L^2}{m^2c^2r_g}\right)^2-\frac{3L^2}{m^2c^2}}=\frac{L^2}{m^2c^2r_g}\left(1\pm\sqrt{1-\frac{3r_g^2m^2c^2}{L^2}}\right).$$

The larger root corresponds to the stable orbit. One can see that

$$1-\frac{3r_g^2m^2c^2}{L^2}>0.$$

Hence

$$-\sqrt{3}mcr_g \le L \le \sqrt{3}mcr_g.$$

Substituting

$$L=\sqrt{3}mcr_g$$

into equation for the radius of circular orbits, we have for the radius of the last stable orbit

$$r_{lso}=3r_g$$

(see the figure below).

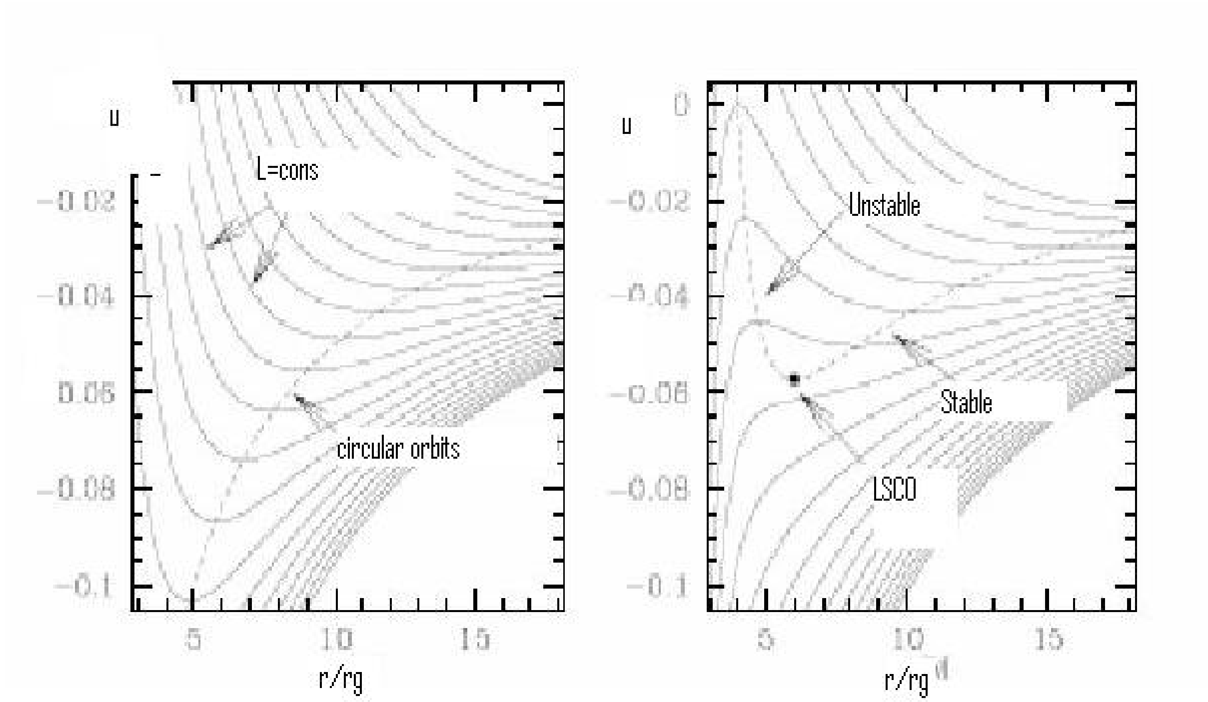
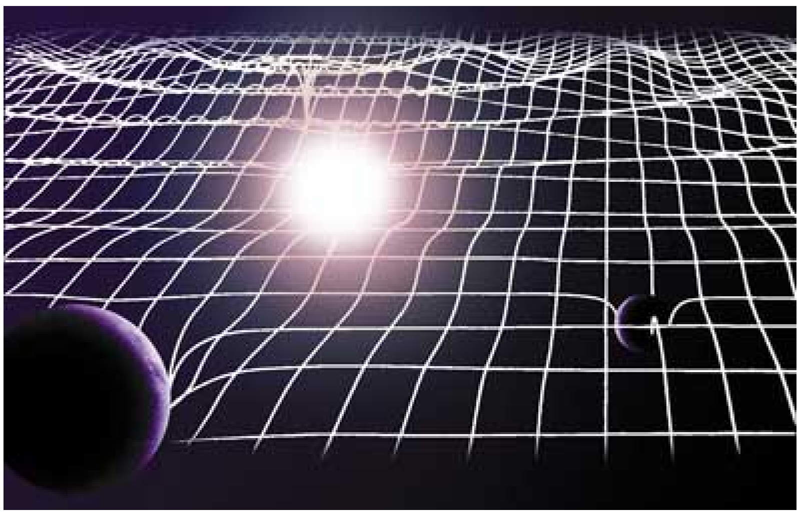


Figure 1: Effective potential energy $u = U_{eff}/mc^2$ for various angular momenta as a function of separation r/r_g of a test mass **m** in orbit about a Newtonian point mass **M** (left panel) and a Schwarzschild black hole of mass **M** (right panel). The solid lines denote contours of constant orbital angular momentum L. Extrema of these contours identify circular orbits, marked by the dashed lines. The circular orbits are stable if the extremum is a minimum, otherwise they are unstable.Abrivation LSCO means the last stable circular orbit

Lecture 6. Part III

GRAVITATIONAL WAVES



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Numerous astronomical observations, as we know from previous lectures, give strong evidence of the existence of black holes, the most interesting and mysterious prediction of general relativity. Black holes are the most powerful sources of gravitational waves.

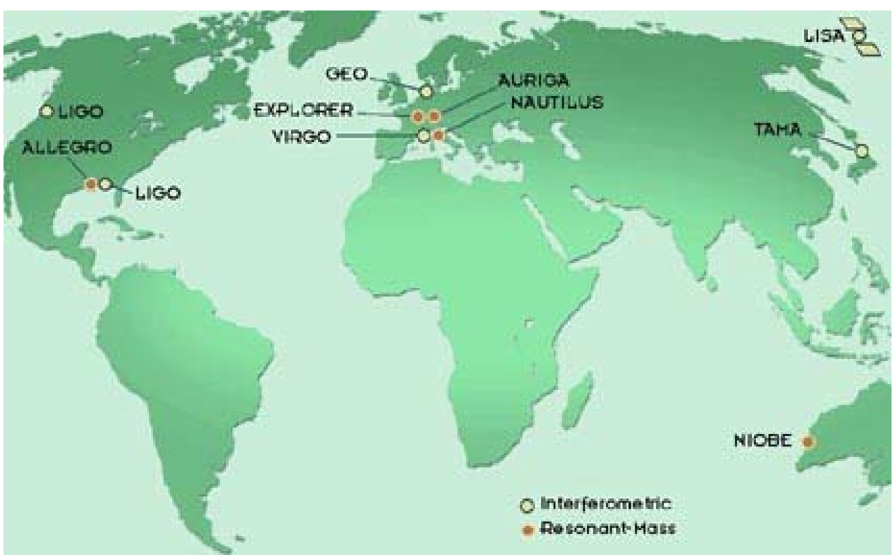
A gravitational wave is a fluctuation in the curvature of spacetime which propagates as a wave, traveling outward from a moving object or system of objects.

Gravitational radiation is the energy transported by these waves. Important examples of systems which emit gravitational waves are binary star systems, where the two stars in the binary are white dwarfs, neutron stars, or black holes. Although gravitational radiation has not yet been directly detected, it has been indirectly shown to exist. This was the basis for the 1993 Nobel Prize in Physics, awarded for measurements of the Hulse-Taylor binary system, considered in part I of this lecture.

1. Gravitational wave astronomy

So far the bulk of information about the Universe comes in form of electromagnetic waves generated by electrically charged particles. Gravitational waves have a totally different nature in comparison with electromagnetic waves, being generated by the motion of massive gravitating objects. and variations of masses of celestial bodies. The observation of gravitational waves will therefore significantly complement the observation of electromagnetic waves (light, radio, micro-waves, X and gamma rays) and of astro-particles (cosmic rays, neutrinos). It will reveal aspects of the Universe not reachable by these means and will extend the observable domain even in the cosmic zones darkened by dust and masked by other phenomena.
The most dramatic processes of the cosmos such as supernova explosions, catastrophic collisions, fusion of binary systems, rotation of pulsars, interaction of black-holes or the original big-bang generate gravitational waves.

Gravitational wave detectors are likely to reveal unsuspected aspects of the Universe. You see below the map with location of gravitational wave detectors which are in operation or in preparation at the present moment:

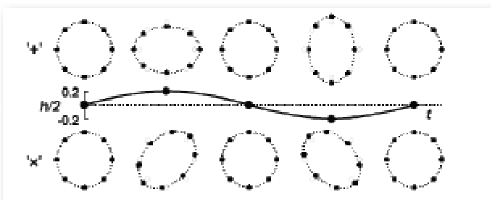


Gravitational waves research in the world

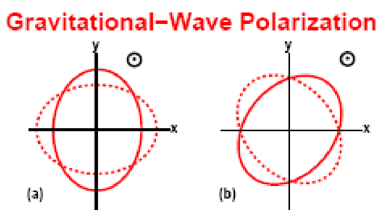
Gravitational wave astronomy, which can be considered as a new window on the Universe, has already started.

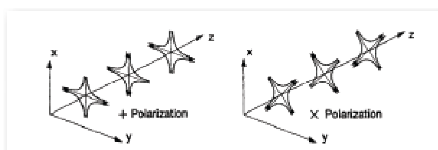
2. Detection of gravitational waves

Gravitational waves distort space-time and produce forces in such a way that the distance between free masses will alternately decrease and increase during the passage of a gravitational wave. An important characteristic is that when there is elongation in one direction there is compression in the perpendicular one. As a result, a circle made of free masses will get successively elongated and contracted in two perpendicular directions.

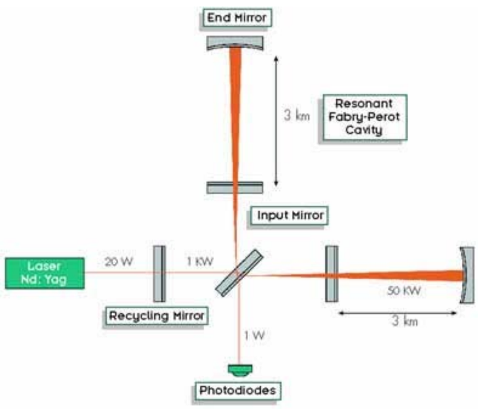


Gravitational waves are polarized:



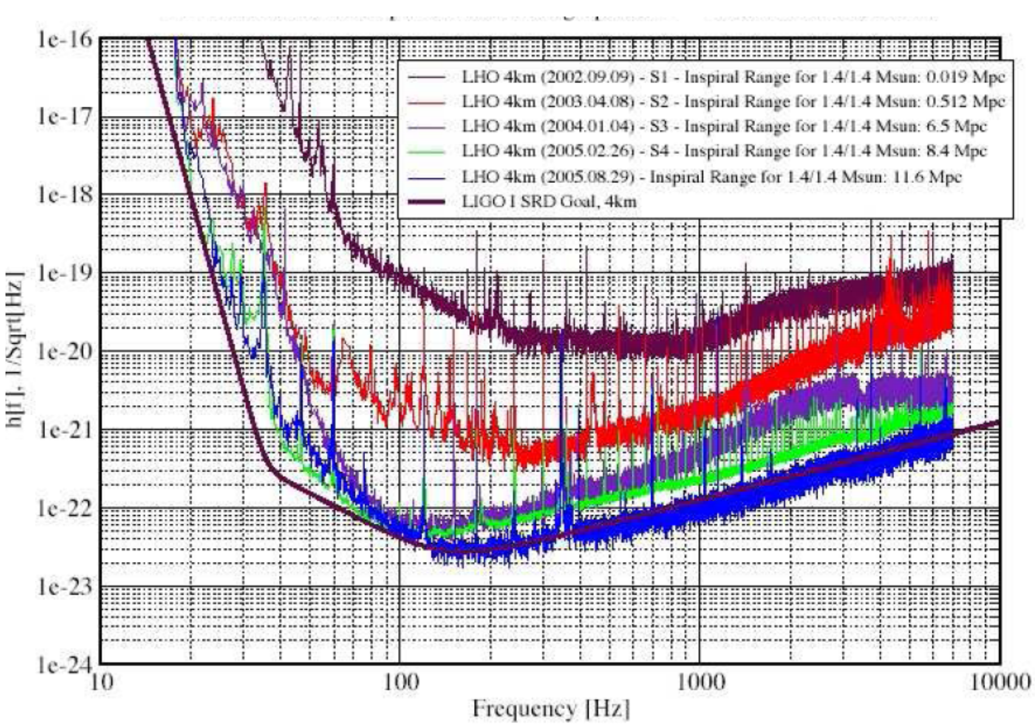


The amplitude of gravitational waves, the dimensionless parameter "h", is measured by the relative variation of distance between two free masses. The absolute variation is therefore proportional to the distance between the two masses. It would typically be one hundred millions times smaller than an atom, however such a small variation of distance can be detected through the phenomenon of interference.
A laser interferometer is very sensitive to differential length variations between its two arms and is ideally suited to the detection of gravitational waves. Because of the extremely high sensitivity required, the length of the arms must be hundreds of kilometers. Since this cannot be practically achieved on earth, one uses multiple reflections between two mirrors to artificially increase the measuring length. Fabry-Perot resonant cavities made of two mirrors are currently employed in gravitational waves interferometers.



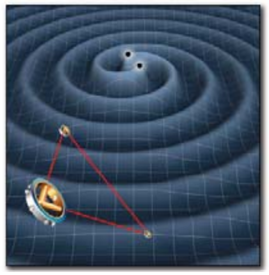
Example 1. L(aser)I(nterferometric)G(ravitational)O(bservatory)

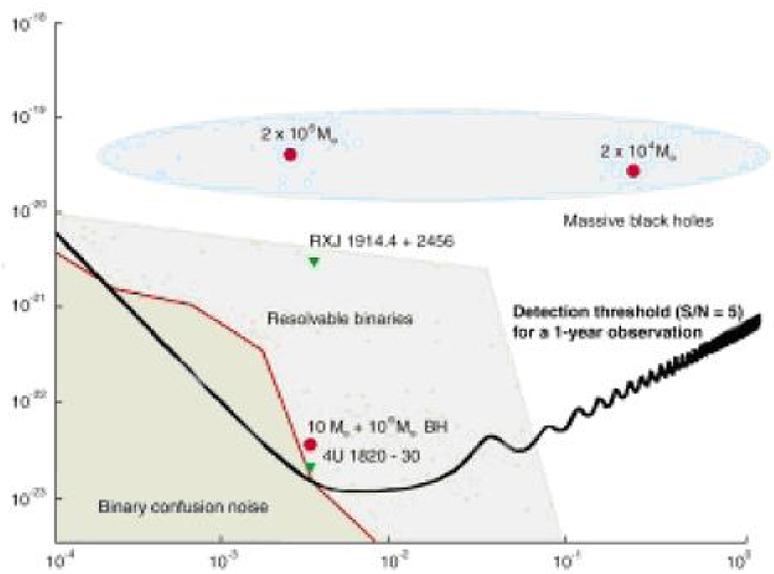
Sensitivities for the LIGO



Example 2. L(aser)I(nterferometric)S(pace)A(ntenna)

Sensitivity for LISA

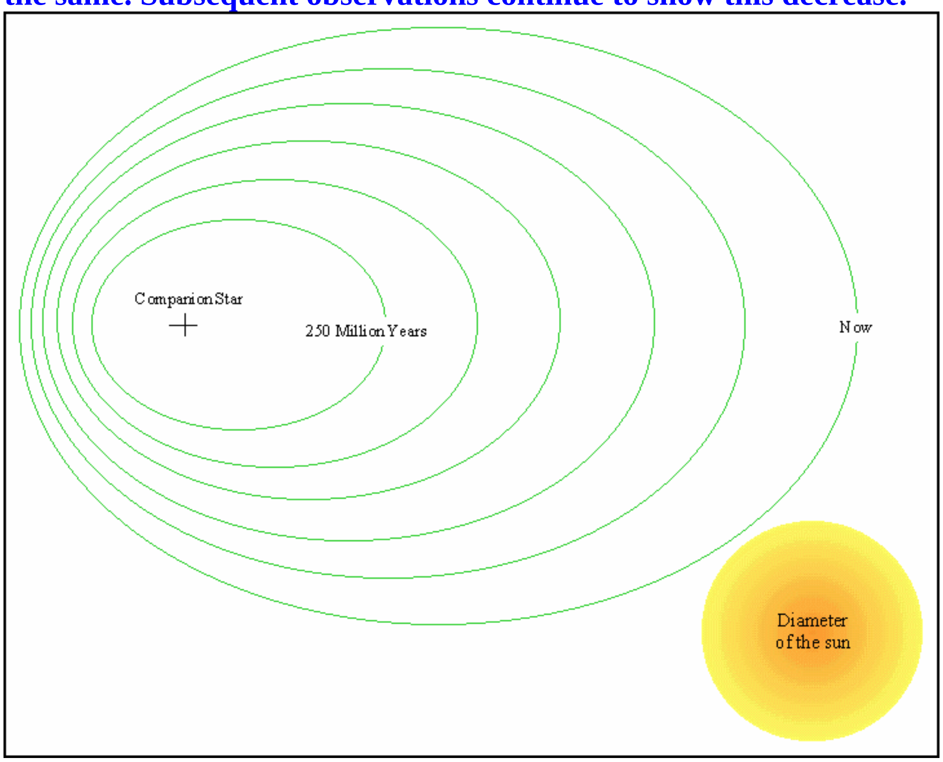




LISA's sensitivity to binary star systems in our Galaxy and black holes in distant galaxies. The heavy black curve shows the LISA detection threshold, giving the noise amplitude of 5 σ after 1-year of observation. At frequencies below 3 mHz, binaries in the Galaxy are so numerous that LISA will not resolve them, and they form a noise background; this is also indicated at its expected 5 s level, coloured dark yellow. In lighter yellow is the region

3. Binary pulsar again

Binary pulsars are one of the few tools scientists have to detect evidence of gravitational waves; Einstein's theory of general relativity predicts that two neutron stars would emit gravitational waves as they orbit a common center of mass, which would carry away orbital energy and cause the two stars to draw closer together. Data collected by Taylor and his colleagues of the orbital period of PRS 1913+16 supported this relativistic prediction; they reported in 1983 that there was a difference in the observed minimum separation of the two pulsars and that expected if the orbital separation had remained constant. In the decade following its discovery the system's orbital period had decreased by about 76 millionths of a second per year-this means that the pulsar was approaching its maximum separation more than a second earlier than it would have if the orbit had remained the same. Subsequent observations continue to show this decrease.



The pulsar's orbit is shrinking with time as shown in this diagram; currently, the orbit shrinks by about 3.1 mm per orbit. The two stars should merge in about 300 million years from now.

4. Propagation of gravitational waves.

A weak gravitational field is a small perturbation of the galilean metric:

$$g_{ik} = \eta_{ik} + h_{ik}.$$

It is easy to show that

$$g^{ik} = \eta^{ik} - \eta^{in}\eta^{km}h_{nk}.$$

The gravitational wave is transverse and traceless part of these perturbations and as we know the plane wave has two independent states of linear polarization.
Using a linear coordinate transformation

$$x'^{i} = x^i + \xi^i,$$

where ξ^i are small functions of x^i. We can impose on h_{ik} the following four supplementary conditions:

$$\eta^{km}h_{mi,k} - \frac{1}{2}\delta_i^k\eta^{nm}h_{nm,k} = 0.$$

After such transformation the Ricci tensor is reduced to

$$R_{ik} = -\frac{1}{2}\eta^{lm}\frac{\partial^2 h_{ik}}{\partial x^l \partial x^m}.$$

According to the Einstein equations in empty space-time

$$R_{ik} = 0,$$

hence gravitational waves satisfy the wave equation

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})h_{ik} = 0,$$

where ∇^2 is the 3-dimensional Laplacian operator.
Consider a ring of test particles initially at rest in the $(y-z)$ plane, perturbed by a plane monochromatic gravitational wave propagating in x-direction with frequency ω and amplitude h_0. Then it is possible to show that all components of h_{ik} can be eliminated by transformation of coordinates except

$$h_{22} = -h_{33} \equiv h_+, \ \ \text{and} h_{23} = h_{32} \equiv h_\times,$$

corresponding to "+" and "×" polarizations. By calculating the physical distances between the test particles on the ring and its center we can determine distortions in shape and in orientation of the ring produced by a gravitational wave at different moments of time and for different polarizations of the gravitational wave:

$$h_+ = h_0 \sin\omega(t - x/c), \ \ h_\times = 0 \ \text{ and } \ h_+ = 0, \ \ h_\times = h_0 \sin\omega(t - x/c).$$

5. Generation of gravitational waves.

Starting from the Einstein equations we can linearize them taking into account that gravitational waves are characterized by small amplitudes. Then in approximation of slow motions and small separations we can use the quadrupole formula for gravitational waves:

$$h_{\alpha\beta} = -\frac{2G}{3c^4R}\frac{d^2D_{\alpha\beta}}{dt^2},$$

where

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})dM \text{ is the quadrupole tensor.}$$

Example

A white dwarf of mass m moves around a black hole of mass $M \gg m$ on a circular orbit with radius r.

Find the frequency of gravitational radiation if T is the orbital period.

Taking into account that

$$x_\alpha = e^\alpha \cos\omega_0 t,$$

where e^α is some constant vector, we have

$$h_{\alpha\beta} \sim \ddot{D}_{\alpha\beta} \sim (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})^{\cdot\cdot} \sim (x_\alpha x_\beta)^{\cdot\cdot}$$

$$\sim e^\alpha e^\beta (\cos^2\omega_0 t)^{\cdot\cdot} \sim \frac{1}{2}e^\alpha e^\beta(1+\cos 2\omega_0 t)^{\cdot\cdot} \sim \cos\omega,$$

where

$$\omega = 2\omega_0 = 4\pi/T.$$

Estimate to an order of magnitude h_0, the amplitude of the gravitational wave.

To an order of magnitude and omitting indices we have

$$h \sim \frac{2G}{3c^4R}\ddot{D} \sim \frac{2G}{3c^4R}\frac{3}{2}(2\omega_0)^2mr^2 \sim \frac{4Gmr^2\omega_0^2}{c^4R} \sim$$

$$\sim \frac{4Gmr^2}{c^4R}\frac{GM}{r^3} \sim \frac{m}{M}\frac{r_g^2}{rR} \sim \frac{m}{M}\frac{r_g}{r}\frac{r_g}{R}.$$