COSMOLOGY ASTM108

PROBLEM SET 2

1. This question is about the conservation equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + \frac{p}{c^2}) = 0 \tag{1}$$

Verify that this equation can be expressed in the form

$$\dot{p}a^3 = \frac{d}{dt}\left[a^3(\rho c^2 + p)\right] \tag{2}$$

and hence that

$$\frac{d}{da}(\rho c^2 a^3) = -3pa^2 \tag{3}$$

Deduce from this expression that whenever the pressure of the matter is positive, its density falls more rapidly than the inverse of the volume of the universe. [**Hint**: It is easier to start with Eq. (2) and show it reduces to Eq. (1). To derive Eq. (3) recall that $\dot{p} = (dp/dt) = (dp/da) \times (da/dt)$].

(In question 5 of Ex. Sheet I, you should have found that this is result holds true when $\gamma > 1$. The above argument shows that it is also true for an arbitrary equation of state, not just the simple linear relationship we have considered).

2. The aim of this and the next question is to introduce the *cosmological constant*, Λ, a term we will meet more than once throughout the course. The cosmological constant may be interpreted in a number of different ways. One is to view it as a special form of 'exotic' matter with an equation of state given by

$$p_\Lambda = -\rho_\Lambda c^2$$

with a constant density, ρ_Λ.
Assuming ρ_Λ to be positive, the Friedmann equation is given by

$$H^2 = \frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho_\Lambda - \frac{kc^2}{a^2} \tag{4}$$

Find the solution to the Friedmann equation (4) – in other words the dependence of the scale factor, $a(t)$, with time – for a universe where $k = 0$.

In the case where $kc^2 = +1$, the solution is given by

$$a(t) = \frac{1}{b}\cosh(bt) \tag{5}$$

where b is a constant. By substituting Eq. (5) into the Friedmann equation (4), determine how this constant is related to the cosmological constant, ρ_Λ.

Sketch the qualitative behaviour of the scale factor as a function of time for both $k = 0$ and $kc^2 = +1$. How do the two solutions compare to one another at late times? What can you deduce about the future destiny of a universe with $kc^2 = +1$ containing only a cosmological constant?
(For this question you may need to refer to the handout on hyperbolic functions. Cosmologies containing just a cosmological constant are known as de Sitter universes, after the relativist who first studied them in 1917).

3. Consider a universe containing a cosmological constant and ordinary, pressureless matter with non–negative density, $\rho > 0$. Show that this universe can be static if and only if the cosmological constant is positive and $k > 0$ in the Friedmann equation. [**Hint**: To be static, the universe must satisfy the conditions $\dot{a} = 0$ and $\ddot{a} = 0$, simultaneously].
(This model is known as Einstein's static universe. It was later shown to be unstable to small perturbations and Einstein later dismissed the cosmological constant as the 'biggest blunder of my life' as he missed the opportunity to predict the expansion of the universe).

4. Consider the Friedmann equation with $kc^2 > 0$ for a universe containing pressureless matter, so that $\rho = \rho_0 a_0/a^3$. Show by direct substitution that the parametric solution

$$a(\theta) = \frac{4\pi G\rho_0}{3kc^2}(1 - \cos\theta), \qquad t(\theta) = \frac{4\pi G\rho_0}{3(kc^2)^{3/2}}(\theta - \sin\theta) \tag{6}$$

solves the Friedmann equation, where $0 \leq \theta \leq 2\pi$. [**Hint**: Recall that $da/dt = (da/d\theta) \times (d\theta/dt) = (da/d\theta) \div (dt/d\theta)$ and evaluate the derivatives from Eq. (6). Also, without loss of generality, you may scale the scale factor of the universe so that its present value is $a_0 = 1$].
Sketch a and t as functions of θ. Describe qualitatively the behaviour of the universe. Sketch a as a function of t.

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