COSMOLOGY ASTM108

PROBLEM SET 3

1. From the definition of the Ω–parameter,

$$\Omega = \frac{8\pi G\rho}{3H^2}$$

where ρ represents the *total* density of all matter in the universe, show that the quantity

$$\rho a^2 \left(1 - \Omega^{-1}\right) = \text{constant} \tag{1}$$

Consider a universe with arbitrary spatial curvature that contains both radiation and matter components. Show that the epoch of matter–radiation equality occurs when

$$(1 - \Omega^{-1})_{\rm eq} = \frac{(1 - \Omega^{-1})_0}{1 + z_{\rm eq}} \tag{2}$$

where $z_{\rm eq}$ is the redshift corresponding to that time. (Hint: assume that the radiation is always negligible after the epoch of matter–radiation equality).
Given that radiation–matter equality occurs when $z_{\rm eq} \approx 6000$, and that the density of matter today is about 30 % that of the critical density, estimate the density of matter relative to that of the critical density at the epoch of matter–radiation equality. Does your answer surprise you?

2. Consider a spatially flat universe dominated entirely by radiation. By deriving an expression for the age of such a universe in terms of the Hubble parameter, calculate how old such a universe would be by the time the Hubble parameter has reduced to a value $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. How does this compare with the age of a flat, pressureless universe with the same value of the Hubble constant?

3. By following a similar method to that taken in the lectures for a closed universe, show that the Friedmann equation for an open (negatively curved) universe containing pressureless matter can be written in the form

$$\dot{a}^2 = \frac{\Omega_0}{H_0\left(1 - \Omega_0\right)^{3/2}} \frac{1}{a} + 1 \tag{3}$$

Hence, deduce that the parametric solution

$$a(\phi) = \frac{\Omega_0}{H_0\left(1 - \Omega_0\right)^{3/2}} \sinh^2\phi, \qquad t(\phi) = \frac{\Omega_0}{H_0\left(1 - \Omega_0\right)^{3/2}} \left(\frac{1}{2}\sinh(2\phi) - \phi\right) \tag{4}$$

satisfies the Friedmann equation (3).
By noting that the present value of the scale factor, a_0, can be expressed in terms of H_0 and Ω_0, verify that the present age of such a universe is given by

$$t_0 = \frac{1}{H_0} \frac{\Omega_0}{(1-\Omega_0)^{3/2}} \left[\frac{(1-\Omega_0)^{1/2}}{\Omega_0} - \sinh^{-1}\left(\sqrt{\frac{1-\Omega_0}{\Omega_0}} \right) \right] \tag{5}$$

Hint: The method is very similar to that we employed in the lectures for a closed universe. You will need the identities:

$\cosh(2x) = 1 + 2\sinh^2 x$
$\sinh 2x = 2\sinh x \cosh x$
$\cosh^2 x - \sinh^2 x = 1.$

4. The aim of this question is to derive the simplified expression quoted in the lectures for the age of the universe when it is nearly spatially flat at the present epoch. To this end, define a parameter ϵ such that

$$\Omega_0 \equiv 1 - \epsilon, \qquad \epsilon \ll 1 \tag{6}$$

Then by employing the Taylor expansions

$$\sinh^{-1} x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots \tag{7}$$

and

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2}x^2 + \ldots \tag{8}$$

for small x in Eq. (5), show that

$$t_0 = \frac{2}{3H_0}\left[1 + \frac{1}{5}(1-\Omega_0)\right] \tag{9}$$

[**Hint**: You need to consider the x^5 term in the expansion of Eq. (7). Although this equation requires some algebra, it provides a good example of how a complicated expression can be approximated to something more illuminating. It also provides good practice of employing Taylor expansions in a real setting].

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