Chapter 3

Special Relativity

'Really this is what is meant by the Fourth Dimension, though some people who talk about the Fourth Dimension do not know they mean it. It is only another way of looking at Time. There is no difference between time and any of the three dimensions of space except that our consciousness moves along it. But some foolish people have got hold of the wrong side of that idea. You have all heard what they have to say about this Fourth Dimension?'

3.1 Minkowski Space-time

In modern terms, special relativity is the study of physics in a universe governed by the Minkowski metric, equation (2.97). The Minkowski metric has coordinates

$$(\mathsf{X}^0, \mathsf{X}^1, \mathsf{X}^2, \mathsf{X}^3) = (ct, x, y, z) \tag{3.1}$$

where t is time and c is the speed of light. Note that the time component, by convention, is distinguished by being given the index 0. Also, the metric is diagonal,

$$\eta_{ab} = Diagonal(1, -1, -1, -1), \tag{3.2}$$

with the time component of opposite sign to the spatial components.

We will examine the geometry of the Minkowski metric in detail. First recall that the spatial part of η_{ab} is Cartesian, apart from an overall minus sign. We could, of course, transform to another coordinate system such as spherical polars. However, this would introduce an apparent dependence of the metric elements on position, which would obscure the simplicity and symmetry of the metric. Thus for this chapter we will exclusively use Cartesian spatial coordinates.

We first note that the Minkowski metric is independent of position and time. This fact gives it the property of

Definition 3.1 *Homogeneity* An object or physical law is homogeneous if it has the same form at all places, i.e. (*its form is invariant to translations*).

A rotation of the spatial (x, y, z) axes leaves the Minkowski metric unchanged. ***Exercise 3.1***

Show that the Minkowski metric is unchanged by a rotation by an angle ψ about the z axis, where

$$t \rightarrow t; \tag{3.3}$$

$$x \rightarrow x\cos\psi - y\sin\psi; \tag{3.4}$$

$$y \rightarrow y\cos\psi + x\sin\psi; \tag{3.5}$$

$$z \rightarrow z. \tag{3.6}$$

Thus it has the property of isotropy.

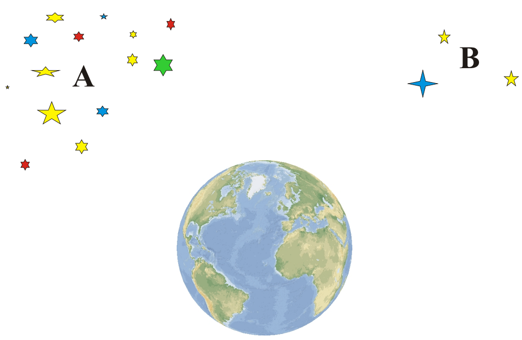
Definition 3.2 *Isotropy* An object or physical law is isotropic if it has the same in all directions, i.e. *its from is invariant to rotations, about any central point and axis.*

Objects can be homogeneous but not isotropic.

Examples:

- A uniform magnetic field. The field looks the same at all points in space, but points in a particular direction.
- A regular crystal. The crystal structure may appear the same at different places, but the molecular bonds are oriented in particular directions
- Most fabrics are woven with a warp and a weft, with the result that their ability to stretch depends on direction.

It is not possible to be isotropic but non-homogeneous. For example, consider a distribution of stars. If the distribution is non-isotropic, more stars are seen in some directions than others. But then regions seen in different directions (A and B) must be different and therefore non-homogeneous.



$$\boxed{\text{Isotropic} \Rightarrow \text{Homogeneous}}. \tag{3.7}$$

Also note that spherical symmetry about some central point does not imply isotropy about all points.

3.2 Units

The zeroth coordinate in Minkowski space-time is $\mathsf{X}^0 = ct$. The presence of the factor of c makes many of the equations more complicated. But we need this factor because traditional units for time and space are different. In order to understand space and time in a unified way, we need to employ a system of units which treats space and time more equally.

Exercise 3.2 *Suppose there were a move to convert the measure of distance on British roads to kilometers. However, this move was fiercely resisted by half of the population. In a political compromise, it was decided to measure East-West distances with kilometers, and North-South distance with miles.*

Imagine coping with this mixed system. What would be the distance from London to Manchester? What would speedometers and odometers look like?

Relativistic Units

In conventional units the speed of light in a vacuum is $c = 2.997\ldots \times 10^8\,\mathrm{m\,s^{-1}}$.

In a relativistic system of units $c = 1$. There are two ways of constructing such systems.

a. Use a basic unit of time; the length unit will be the distance travelled by light in that time.

   - **A)** choose the time unit to be the second (s), and define the unit of length to be the *light-second (ℓs)*

$$1\ell\,\mathrm{s} = 3 \times 10^8\,\mathrm{m}. \tag{3.8}$$

In these units, $c = 1\ell\,\mathrm{s}/\,\mathrm{s}$. Usually, we do not bother writing the $\ell\,\mathrm{s}/\,\mathrm{s}$, and so $c = 1$.

- **B**) *Time unit: year (y); length unit:* light-year (ℓ y).

$$1\,\ell\,\mathrm{y} \approx 3\times 10^{8}\,\frac{\mathrm{m}}{\mathrm{s}}\cdot 3.4\times 10^{7}\,\mathrm{s} \qquad (3.9)$$

$$\approx 10^{16}\,\mathrm{m} \qquad (3.10)$$

and $c = 1\,\ell\,\mathrm{y}/\,\mathrm{y}$. Again we will ignore the $\ell\,\mathrm{y}/\,\mathrm{y}$ and just say $c = 1$.

b. Use a basic unit of length; the time unit will be the interval of time needed for light to travel that distance.

- Choose the length unit to be the metre (m), and the time unit to be the *light-metre* (ℓ m)

$$1\ell\,\mathrm{m} \approx 3\times 10^{-9}\,\mathrm{s} \qquad (3.11)$$

and $c = 1\,\mathrm{m}/\ell\,\mathrm{m} = 1$.

Example 3.1 Express Watts in relativistic units with basic units second, kilogram.
Solution

$$1W = 1\,\mathrm{kg\ m^2\ s^{-3}} \qquad (3.12)$$

$$= 1\,\mathrm{kg\ s^{-1}}\,\frac{\mathrm{m^2}}{\mathrm{s^2}}\left(\frac{l\,\mathrm{s}}{3\times 10^{8}\,\mathrm{m}}\right)^2 \qquad (3.13)$$

$$= \frac{1}{9\times 10^{16}}\,\mathrm{kg\ s^{-1}}\left(\frac{l\,\mathrm{s}}{\mathrm{s}}\right)^2 \qquad (3.14)$$

$$\approx 1.1\times 10^{-17}\,\mathrm{kg\ s^{-1}} \qquad (3.15)$$

N.B. We could have reached this by multiplying by c or c^{-1} until only the units kg and s were left (cancel out as many factors of c as necessary to get the units right).
In reverse: what is $1kg\ s^{-1}$ in Watts?
Solution Multiply by c^2 to obtain the right units

$$\Rightarrow 1\,\mathrm{kg\ s^{-1}} = 9\times 10^{16}\,\mathrm{kg\ m^2\ s^{-3}} \qquad (3.16)$$

$$= 9\times 10^{16}W. \qquad (3.17)$$

Exercise 3.3 *The gravitational constant is* $G = 6.67\,10^{-8}\mathrm{cm^3 g^{-1} s^{-2}}$. *Express* G *in relativistic units, with the basic units being grams and centimetres (time measured in light-cms). Next in relativistic units calculate the escape velocity* V *from the surface of the earth. Also calculate* $\gamma - 1$. *(Recall that the gravitational potential energy of an object of mass m at the surface of the Earth is* $-GM_{\oplus}m/R_{\oplus}$, *where* $M_{\oplus}$ *is the mass of the Earth and* $R_{\oplus}$ *is its radius.)*
Earth mass: $M_{\oplus} = 6\,10^{27}$ *g.*
Earth radius: $R_{\oplus} = 6.4\,10^{3}$ *km.*

Exercise 3.4 *The acceleration due to gravity at the Earth's surface is* $1\mathrm{g} = 9.8\frac{\mathrm{m}}{\mathrm{s}^2}$. *Express this in relativistic units with basic unit being the year (i.e. lengths are measured in light years). (1 year* $\approx 3.2\times 10^{7}s$.*)*

3.3 Einstein's Axioms of Special Relativity

The Minkowski metric, as we have seen, is invariant to translations and spatial rotations. However, in a four dimensional space-time manifold, we can also consider rotations involving both space and time coordinates. Such mixing of space and time coordinates may seem mysterious, but actually the effect is simple: the spatial origin $x = y = z = 0$ in the rotated system moves at a constant velocity with respect to the original system. This is called *boost*.

Definition 3.3 *Boost* A boost is a transformation to a coordinate system moving at a constant relative velocity with respect to the original system.

Einstein's famous 1905 paper demonstrated that we needed a new conception of space-time if we were to have a theory of electromagnetism which looked the same in all coordinate systems, especially ones reached via a boost transformation. He started out with the idea of a coordinate system, or *reference frame* in which there are no inertial forces such as centrifugal or Coriolis forces.

Definition 3.4 *Inertial Reference Frame* An inertial reference frame (IRF) is a co-ordinate system for space-time with Cartesian spatial co-ordinates, and where there exist no inertial (fictitious) forces.

The invariance of Maxwell's equations led Einstein to believe that the speed of light does not change when boosting to a moving reference frame. He wrote down two axioms for physics in general, and electromagnetic radiation in particular:

a. The laws of physics are invariant to translations, rotations and boosts.

b. The speed of light is the same in all IRF's.

3.4 Space-Time Diagrams

Space-time diagrams place time on the vertical axis, with one space dimension on the horizontal axis (or two in a horizontal plane).

Definition 3.5 *World–line* The *world–line* of an object is the path it traces in space-time.

If we just look at one space dimension (x), the velocity of the object is $\mathrm{d}x/\mathrm{d}t$. Since the vertical axis is time, however, the slope of the world line is $\mathrm{d}t/\mathrm{d}x$.

For photons $c = \mathrm{d}x/\mathrm{d}t = 1$. In special relativity light moves at an angle of $\arctan(1) = \pi/4$ on a space-time diagram.

Choose a point on an object's world-line, P, to be $\tau = 0$. Then let τ be the arc–length away from P,

$$\tau = \int_P^Q \sqrt{g_{ab}\,\mathrm{d}\mathsf{X}^a\,\mathrm{d}\mathsf{X}^b}. \tag{3.18}$$

Definition 3.6 *Proper Time* The arc–length τ along a world–line is called the *proper time*.

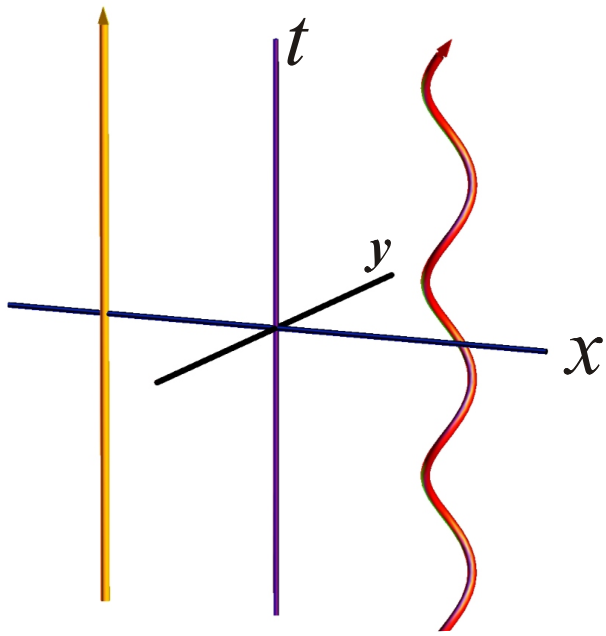
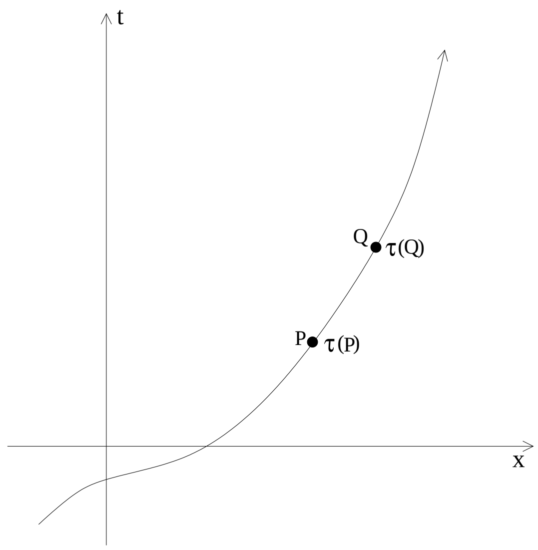


Figure 3.1: The object with the world–line to the left is at rest. The object to the right is moving in a circular orbit.



Definition 3.7 *Event* A space-time event $\mathcal{P}$ is a point in space-time.

Suppose at event $\mathcal{P}$ a camera flash goes off, sending an expanding sphere of light into space (and time). We can most easily picture this expanding sphere by suppressing one dimension. For example, suppose the z co-ordinate of $\mathcal{P}$ is 0. We can then consider how the light travels in the $z = 0$ plane. The expanding sphere of light intersects this plane as an expanding circle. In a space-time diagram showing the x, y, and t directions, the expanding circle traces out a cone.

The future light cone at an event $\mathcal{P}$ shows how a pulse of light emitted at $\mathcal{P}$ travels through space-time. A light-cone splits space-time into space-like and time-like parts. Objects moving slower than light (i.e. all massive objects!) can only reach the time-like parts. One would need to move faster than light to reach the space-like parts.

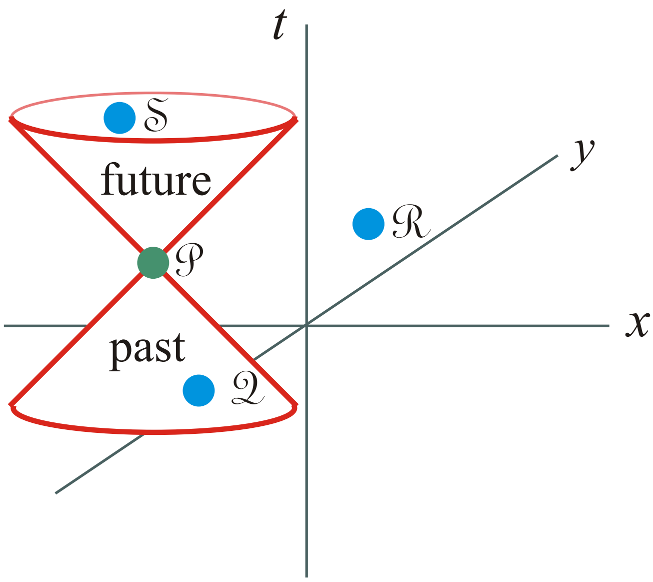


Figure 3.2: The interval $\mathcal{PQ}$ is light-like ($\Delta\tau = 0$), as $\mathcal{Q}$ is on $\mathcal{P}$'s past light cone. The interval $\mathcal{PR}$ is space-like ($\Delta\tau < 0$), while $\mathcal{PS}$ is time-like ($\Delta\tau > 0$).

A particle with 3-velocity $\vec{\mathbf{V}} = (\,\mathrm{d}x/\,\mathrm{d}t,\ \mathrm{d}y/\,\mathrm{d}t,\ \mathrm{d}z/\,\mathrm{d}t)$ satisfies

$$\left|\vec{\mathbf{V}}\right|^2 \mathrm{d}t^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{3.19}$$

For a photon, $\left|\vec{\mathbf{V}}\right|^2 = c = 1$, and so $\mathrm{d}t^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ between any two events along a photon's world-line. By Einstein's second axiom, this is true for a photon as seen in *any* IRF.

Suppose we define a small interval between two points $d\tau$ by

$$\mathrm{d}\tau^2 \equiv \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2. \tag{3.20}$$

Then

$$\mathrm{d}\tau^2 = 0 \tag{3.21}$$

along a photon path in all IRFs. This suggests that $\mathrm{d}\tau$ behaves like a metric line element. In fact, it is precisely the line element

$$\mathrm{d}\tau^2 = g_{ab}\,\mathrm{d}X^a\,\mathrm{d}X^b \tag{3.22}$$

resulting from the Minkowski metric

$$g_{ab} = \eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.23}$$

Space-time equipped with η_{ab} is called "Minkowski space" or M^4. A world-line is a curve in M^4.

Theorem

Proper time equals clock time in the rest frame of the object.
Proof: In the rest frame R,

$$\mathrm{d}\mathsf{R}^1 = \mathrm{d}\mathsf{R}^2 = \mathrm{d}\mathsf{R}^3 = 0, \tag{3.24}$$

so $\mathrm{d}\tau^2 = \mathrm{d}\mathsf{R}^{0\,2} = \mathrm{d}t_{\mathsf{R}}^2$. We generally chose to have coordinate time increase in the same direction as proper time, so we can take the positive square root:

$$\boxed{\mathrm{d}\tau = \mathrm{d}t_{\mathsf{R}}.} \tag{3.25}$$

The tangent vector to the world line is the 4-vector

$$U^a = \frac{\mathrm{d}\mathsf{X}^a}{\mathrm{d}\tau} = \begin{pmatrix} \mathrm{d}t/\mathrm{d}\tau \\ \mathrm{d}x/\mathrm{d}\tau \\ \mathrm{d}y/\mathrm{d}\tau \\ \mathrm{d}z/\mathrm{d}\tau \end{pmatrix} \tag{3.26}$$

What is the corresponding form U_a?

$$U_a = \eta_{ab}U^b \tag{3.27}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{d}t/\mathrm{d}\tau \\ \mathrm{d}x/\mathrm{d}\tau \\ \mathrm{d}y/\mathrm{d}\tau \\ \mathrm{d}z/\mathrm{d}\tau \end{pmatrix} \tag{3.28}$$

$$= \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, -\frac{\mathrm{d}x}{\mathrm{d}\tau}, -\frac{\mathrm{d}y}{\mathrm{d}\tau}, -\frac{\mathrm{d}z}{\mathrm{d}\tau}\right). \tag{3.29}$$

$$|U|^2 = U_a U^a \tag{3.30}$$

$$= \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 - \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \tag{3.31}$$

$$= \frac{\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2}{\mathrm{d}\tau^2} \tag{3.32}$$

$$= \frac{\mathrm{d}s^2}{\mathrm{d}\tau^2} = \left(\frac{\mathrm{d}\tau}{\mathrm{d}\tau}\right)^2 = 1 \tag{3.33}$$

$$\Rightarrow U_a U^a = 1 \qquad \text{in } \textit{all} \text{ reference frames} \tag{3.34}$$

This is easiest to see in the rest frame, where $U_a = (1, 0, 0, 0)$.

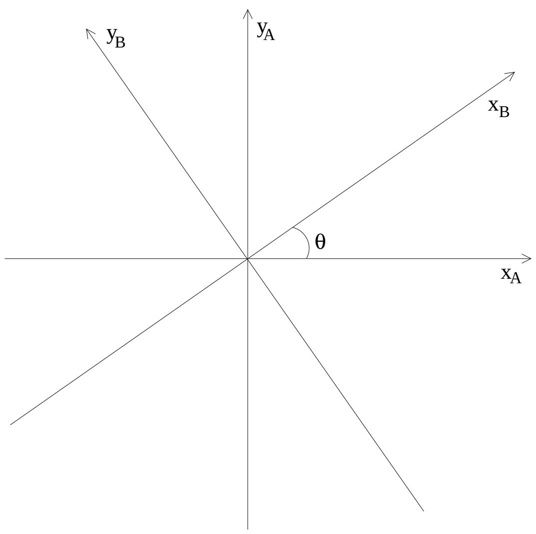
Exercise 3.5 *Suppose a spaceship moves at speed* $\overrightarrow{\mathbf{V}}$ *in the Earth frame. What is* $\mathrm{d}t_E/\mathrm{d}\tau$ *where* t_E *is the Earth time and* τ *is proper time inside the spaceship? Next suppose the spaceship is investigating some scalar function of position* $f(t_E, x_E, y_E, z_E)$ *(e.g. temperature of the interplanetary medium). The ship measures and records* $f(\tau)$ *as it travels through space. Find* $\mathrm{d}f/\mathrm{d}\tau$ *in terms of* $\frac{\partial f}{\partial t_E}$ *and the spatial gradient* $(\frac{\partial f}{\partial x_E}, \frac{\partial f}{\partial y_E}, \frac{\partial f}{\partial z_E})$*. Can you express your result in 4-vector notation?*

3.5 The Poincaré and Lorentz Groups

These are the sets of transforms from one IRF to another i.e. that preserve $g_{ab} = \eta_{ab}$. Thus going from coordinates X_{A} to X_{B} with transform $L_{\mathsf{A}\to\mathsf{B}}$ gives

$$g_{\mathsf{A}ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \qquad g_{\mathsf{B}ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.35}$$

Example 3.2 Rotation about Axes.



$L_{\mathsf{A}\to\mathsf{B}}$ has the rule

$$t_{\mathsf{B}} = t_{\mathsf{A}} \tag{3.36}$$
$$x_{\mathsf{B}} = \cos\theta x_{\mathsf{A}} + \sin\theta y_{\mathsf{A}} \tag{3.37}$$
$$y_{\mathsf{B}} = -\sin\theta x_{\mathsf{A}} + \cos\theta y_{\mathsf{A}} \tag{3.38}$$
$$z_{\mathsf{B}} = z_{\mathsf{A}} \tag{3.39}$$

$$\Rightarrow L_{\mathsf{A}\to\mathsf{B}} = \frac{\partial \mathsf{B}^a}{\partial \mathsf{A}^b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.40}$$

Example 3.3 Translations.
Consider the transformation $P_{\mathsf{A}\to\mathsf{B}}$

$$\begin{matrix} t_{\mathsf{B}} = t_{\mathsf{A}} + 3 & y_{\mathsf{B}} = y_{\mathsf{A}} - 5 \\ x_{\mathsf{B}} = x_{\mathsf{A}} - 2 & z_{\mathsf{B}} = z_{\mathsf{A}} - 4 \end{matrix} \tag{3.41}$$

The orientation of the axes does not change: $\frac{\partial \mathsf{B}}{\partial \mathsf{A}} = \underline{I}^4$. However, the origin moves – the origin of the B system is at $(t_a, x_a) = (5, 4)$.

The set of *all* η_{ab} preserving transformations is called the *Poincaré group*, while those which leave the origin fixed (no translation, just rotation) are the *Lorentz group*. The Lorentz group is a subgroup of the Poincaré group.

3.5.1 Group Axioms

a. Closure:

$$\begin{aligned} & X_{\mathsf{A}} \xrightarrow{P_{\mathsf{A}\to\mathsf{B}}} X_{\mathsf{B}} \xrightarrow{P_{\mathsf{B}\to\mathsf{C}}} X_{\mathsf{C}} && (3.42)\\ = \; & X_{\mathsf{A}} \xrightarrow{P_{\mathsf{A}\to\mathsf{C}}} X_{\mathsf{C}}. && (3.43)\end{aligned}$$

Let $P_{\mathsf{A}\to\mathsf{B}}$ and $P_{\mathsf{B}\to\mathsf{C}}$ be *any* elements of the Poincaré group. Then: $P_{\mathsf{A}\to\mathsf{C}} = P_{\mathsf{B}\to\mathsf{C}}P_{\mathsf{A}\to\mathsf{B}}$ (composition of the two transformations) preserves η_{ab} if *both* $P_{\mathsf{A}\to\mathsf{B}}$ and $P_{\mathsf{B}\to\mathsf{C}}$ do.

b. Identity:
If $X_{\mathsf{A}} = X_{\mathsf{B}}$, then

$$P_{\mathsf{A}\to\mathsf{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.44}$$

c. Inverse:

$$P^{-1}_{\mathsf{A}\to\mathsf{B}} = P_{\mathsf{B}\to\mathsf{A}} \tag{3.45}$$

d. Associative:

$$\left(P_{\mathsf{A}\to\mathsf{B}}P_{\mathsf{B}\to\mathsf{C}}\right) P_{\mathsf{C}\to\mathsf{D}} = P_{\mathsf{A}\to\mathsf{B}} \left(P_{\mathsf{B}\to\mathsf{C}}P_{\mathsf{C}\to\mathsf{D}}\right) \tag{3.46}$$

Theorem:
For any Lorentz transform $L_{\mathsf{A}\to\mathsf{B}}$

$$|det(L_{\mathsf{A}\to\mathsf{B}})| = 1 \tag{3.47}$$

Proof:
Both A and B are inertial frames, so $g_{\mathsf{A}ab} = \eta_{ab}$ and $g_{\mathsf{B}ab} = \eta_{ab}$. Thus

$$\begin{aligned}\eta_{ab} &= \frac{\partial \mathsf{B}^c}{\partial \mathsf{A}^a}\frac{\partial \mathsf{B}^d}{\partial \mathsf{A}^b}\,\eta_{cd} && (3.48)\\ &= L^c{}_a L^d{}_b\,\eta_{cd}. && (3.49)\end{aligned}$$

where $L = L_{\mathsf{A}\to\mathsf{B}}$. Now, the determinant of the product of two matrices is the product of the determinants, so

$$\begin{aligned}\det(\eta) &= \det(L)^2\det(\eta) && (3.50)\\ \implies \det(L)^2 &= 1. && (3.51)\end{aligned}$$

$$\therefore \boxed{|det(L_{\mathsf{A}\to\mathsf{B}})| = 1}. \tag{3.52}$$

Definition 3.8 *Proper and Improper Transforms*

- 'Proper' Lorentz transforms have $\det(L) = 1$
- 'Improper' Lorentz transforms have $\det(L) = -1$

Example 3.4 Mirror Transform

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}_{\mathsf{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}_{\mathsf{A}} . \tag{3.53}$$

This improper transform reflects objects in the x direction.

3.6 Lorentz Boosts

3.6.1 Deriving the transformation matrix

Suppose a spaceship moves, velocity $v\hat{i}$ w.r.t. Earth:

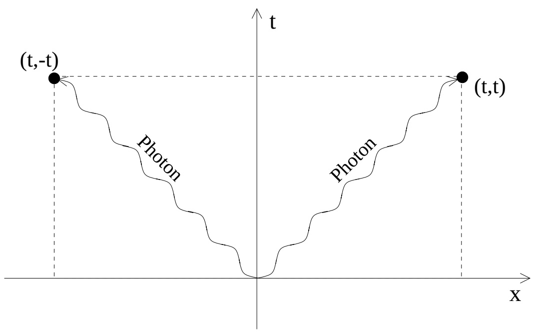
$$\text{Ship's rest frame: } \mathsf{S} \tag{3.54}$$
$$\text{Earth's rest frame: } \mathsf{E} \tag{3.55}$$

Assume the origins coincide – $\mathsf{S}^a = 0$ is the same event as $\mathsf{E}^a = 0$. There will be many Lorentz transformations which go from the Earth frame to the Ship's frame; these will differ by rotations in space. We can guess, however, that there will be a simple one where the y and z coordinates do not change: $y_S = y_E$ and $z_S = z_E$. Thus we will try transforms of the form:

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}_S = \begin{pmatrix} ? & 0 & 0 & 0 \\ 0 & ? & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}_E \tag{3.56}$$

$$\implies \begin{pmatrix} t \\ x \end{pmatrix}_S = \begin{pmatrix} \gamma & \delta \\ \mu & \nu \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}_E \tag{3.57}$$

1) <u>$c = 1$ in both frames</u>



In all frames, a photon moving to the right passes through events with

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} . \tag{3.58}$$

A photon moving to the left, on the other hand, passes through

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t \\ -t \end{pmatrix} . \tag{3.59}$$

Suppose in the Earth frame a photon passes through the event

$$\begin{pmatrix} t \\ x \end{pmatrix}_E = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_E .$$

In ship's co-ordinates,

$$\begin{pmatrix} t \\ x \end{pmatrix}_S = \begin{pmatrix} t \\ t \end{pmatrix}_S = \begin{pmatrix} \gamma & \delta \\ \mu & \nu \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_E \tag{3.60}$$

Thus

$$\begin{aligned} t_S &= \gamma + \delta && (3.61) \\ &= \mu + \nu && (3.62) \\ \Rightarrow \gamma + \delta &= \mu + \nu. && (3.63) \end{aligned}$$

A photon could also pass through $\begin{pmatrix} t \\ x \end{pmatrix}_E = \begin{pmatrix} 1 \\ -1 \end{pmatrix}_E$, which has ship's co-ordinates

$$\begin{aligned} \begin{pmatrix} t \\ x \end{pmatrix}_S &= \begin{pmatrix} \gamma & \delta \\ \mu & \nu \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}_E && (3.64) \\ &= \begin{pmatrix} t \\ -t \end{pmatrix}_S && (3.65) \\ \Rightarrow t_S &= \gamma - \delta && (3.66) \\ &= \nu - \mu && (3.67) \\ \Rightarrow \gamma - \delta &= \mu - \nu && (3.68) \end{aligned}$$

Combining these two results gives

$$\begin{aligned} &\gamma + \delta = \mu + \nu && (3.69) \\ &\gamma - \delta = \nu - \mu && (3.70) \\ &\Rightarrow 2\gamma = 2\nu && (3.71) \end{aligned}$$

$$\Rightarrow \boxed{\nu = \gamma}, \quad \boxed{\mu = \delta} \tag{3.72}$$

We now have

$$\begin{pmatrix} t \\ x \end{pmatrix}_S = \begin{pmatrix} \gamma & \delta \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}_E . \tag{3.73}$$

2) Follow spatial origin in ship's co-ordinates

On the ship, $\begin{pmatrix} t \\ x \end{pmatrix}_S = \begin{pmatrix} t \\ 0 \end{pmatrix}_S$. But, Earthlings see this move at speed V

$$\Rightarrow \begin{pmatrix} t \\ x \end{pmatrix}_E = \begin{pmatrix} t \\ Vt \end{pmatrix} \tag{3.74}$$

$$\Rightarrow \begin{pmatrix} t \\ x \end{pmatrix}_S = \begin{pmatrix} t \\ 0 \end{pmatrix}_S \tag{3.75}$$

$$= \begin{pmatrix} \gamma & \delta \\ \delta & \gamma \end{pmatrix} \begin{pmatrix} t \\ Vt \end{pmatrix}_E \tag{3.76}$$

And so,

$$t_S = \gamma t_E + \delta V t_E \tag{3.77}$$

$$= (\gamma + \delta V)\, t_E \tag{3.78}$$

$$x_S = \delta t_E + \gamma V t_E \tag{3.79}$$

$$= (\delta + \gamma V)\, t_E \tag{3.80}$$

$$= 0 \tag{3.81}$$

$$\Rightarrow \delta + \gamma V = 0 \tag{3.82}$$

$$\Rightarrow \delta = -\gamma V \tag{3.83}$$

Thus, we have

$$\boxed{\begin{pmatrix} t \\ x \end{pmatrix}_S = \gamma \begin{pmatrix} 1 & -V \\ -V & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}_E}. \tag{3.84}$$

3) <u>Apply $\det(L) = 1$</u>

$$\det \begin{pmatrix} \gamma & -\gamma V \\ -\gamma V & \gamma \end{pmatrix} = 1 \tag{3.85}$$

$$= \gamma^2 - \gamma^2 V^2 \tag{3.86}$$

$$= \gamma^2 \left(1 - V^2\right) \tag{3.87}$$

$$\Rightarrow \gamma^2 = \frac{1}{1 - V^2} \tag{3.88}$$

$$\therefore \boxed{\gamma = \frac{1}{\sqrt{1 - V^2}}}. \tag{3.89}$$

4) <u>Inverse Transformation</u>
From the ship's frame to the Earth's frame

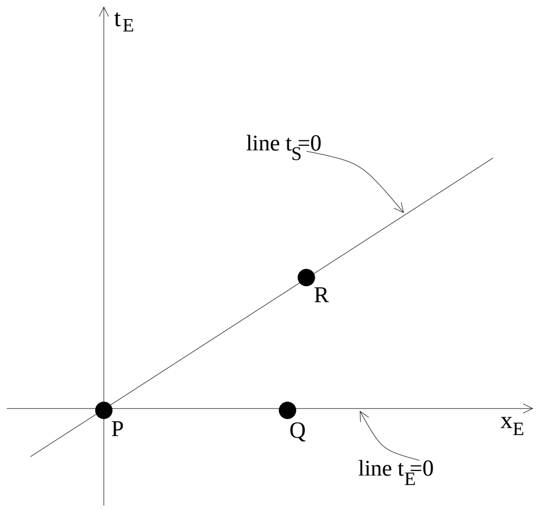
$$\begin{pmatrix} t \\ x \end{pmatrix}_E = \begin{pmatrix} \gamma & \gamma V \\ \gamma V & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}_S \tag{3.90}$$

This is the inverse transform to the one from Earth to ship, as $V \to -V$.

3.7 Simultaneity

A 'surface of simultaneity' is a set of points where t =constant in some reference frame. Lets look at the Ship's surface of simultaneity in the Earth's co-ordinate frame.

The line $t_S = 0$ contains events occurring simultaneously in the ship's frame.



From the inverse transformation,

$$t_E = \gamma V x_S \tag{3.91}$$
$$x_E = \gamma x_S \tag{3.92}$$
$$\Rightarrow t_E = V x_E \,. \tag{3.93}$$

This is the line in the Earth's frame of reference corresponding to $t_S = 0$.

Events P, Q are simultaneous in the Earth's frame, but *not* in the Ship's. Events P, R are simultaneous in the Ship's frame, but *not* in the Earth's.

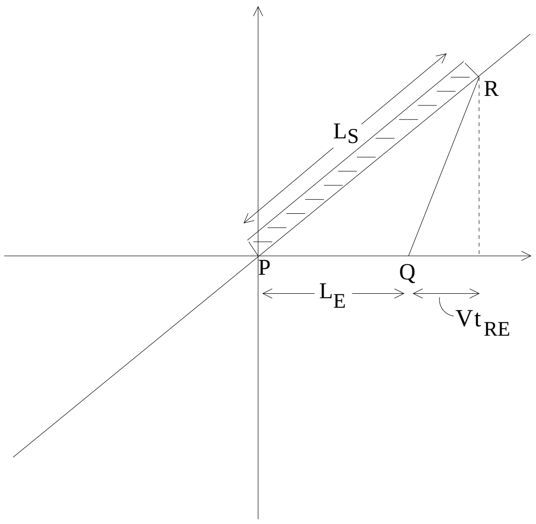
Exercise 3.6 *Suppose a spaceship moves at speed $V\,\hat{x}$ with respect to the Earth, where $V = 1/2$. Let the coordinates in the rest frame of the ship be*

$$\mathsf{S}^a = (\mathsf{S}^0, \mathsf{S}^1) = (t_S, x_S), \tag{3.94}$$

(ignoring the y and z components). Similarly, let $\mathsf{E}^a = (\mathsf{E}^0, \mathsf{E}^1) = (t_E, x_E)$ be coordinates in the rest frame of the Earth. Also let $(0,0)_S = (0,0)_E$. Draw a space-time diagram where the horizontal axis gives x_E and the vertical axis gives t_E. On this diagram draw the line $x_S = 0$ (the time axis in the ship rest frame) and the line $t_S = 0$ (the space axis in the ship rest frame). What is the angle between these lines?

3.8 Length Contraction

noindent**Definition** *Length:* The length of an object is the spatial distance between the ends, measured simultaneously in some reference frame.



Consider a metre stick at rest on the spaceship; The space travellers measure the position of the ends of the stick simultaneously at P, R. Earthlings see P, Q as simultaneous events corresponding to the ends of the stick at $t_E = 0$.

Thus,

	t_E	x_E	t_S	x_S
P	0	0	0	0
Q	0	L_E	?	?
R	?	?	0	L_S

Apply the Lorentz transform to find the ?'s.

$$Q: \quad \begin{pmatrix} t_Q \\ x_Q \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma V \\ -\gamma V & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ L_E \end{pmatrix} = \gamma \begin{pmatrix} -VL_E \\ L_E \end{pmatrix} \tag{3.95}$$

$$R: \quad \begin{pmatrix} t_R \\ x_R \end{pmatrix} = \begin{pmatrix} \gamma & \gamma V \\ \gamma V & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ L_S \end{pmatrix} = \gamma \begin{pmatrix} -VL_S \\ L_S \end{pmatrix} \tag{3.96}$$

The right end of the stick moves from $Q \to R$ with speed V w.r.t. Earth

$$\Rightarrow x_{RE} = x_{QE} + V(t_{RE} - t_{QE}) \tag{3.97}$$

$$= x_{QE} + Vt_{RE} \tag{3.98}$$

If we combine this with the Lorentz transform, we get

$$\gamma L_S = L_E + V(\gamma V L_S) \tag{3.99}$$

$$\Longrightarrow L_E = L_S(\gamma - V^2\gamma) \tag{3.100}$$

$$= L_S\gamma(1 - V^2) \tag{3.101}$$

$$= L_S\gamma\gamma^{-2} \tag{3.102}$$

$$= L_S\gamma^{-1} \tag{3.103}$$

$$\therefore \boxed{L_E = L_S\gamma^{-1}} \tag{3.104}$$

and since $\gamma > 1$,

$$\boxed{L_E < L_S}. \tag{3.105}$$

3.9 Relativistic Dynamics

3.9.1 The 4-momentum

The three-momentum of an object is defined as $\overrightarrow{\mathbf{p}} = m\overrightarrow{\mathbf{V}}$. In space-time, we use $\overline{\mathbf{U}}$ instead of $\overrightarrow{\mathbf{V}}$. For an object travelling at speed $\overrightarrow{\mathbf{V}}$, a Lorentz transformation from the rest frame of the object gives

$$U^a = \begin{pmatrix} \gamma \\ \gamma\overrightarrow{\mathbf{V}} \end{pmatrix}. \tag{3.106}$$

We can check that $|\overline{\mathbf{U}}|^2 = 1$:

$$U_a U^a = (\gamma, -\overrightarrow{\mathbf{V}}) \begin{pmatrix} \gamma \\ \gamma\overrightarrow{\mathbf{V}} \end{pmatrix} = \gamma^2(1 - V^2) = 1. \tag{3.107}$$

Next, we extend momentum from the three dimensional $\overrightarrow{\mathbf{p}}$ to a four-dimensional object. We do this by including the energy E. This makes intuitive sense: classical mechanics conserves E as well as the three components of momentum. Another rational for combining energy and momentum follows from the symmetries of space-time. Noether's theorem (see chapter 7) shows that momentum conservation is a direct consequence of the homogeneity of space (i.e. invariance to spatial translations). But Noether also showed that energy conservation follows in the same way from the homogeneity of time (invariance to time translations). Thus combining space and time into space-time corresponds to combining energy and momentum into one object as well.

As we will see later, when analyzing orbits, the 4-momentum works most naturally as a form rather than a vector. In other areas of physics momentum also appears as dual or conjugate to vectors, in particular the position vector. Recall, for example, that in quantum theory momentum appears paired with the position vector $\overrightarrow{\mathbf{x}}$, e.g. in the Fourier transform term $\exp(i\overrightarrow{\mathbf{p}} \cdot \overrightarrow{\mathbf{x}}/\hbar)$. In these cases $\overrightarrow{\mathbf{p}}$ combines with $\overrightarrow{\mathbf{x}}$ to form a scalar, just as a form combines with a vector to form a scalar. This is why we will begin with its definition as a form. We are free to give names to the components of $\underline{\mathbf{p}}$. In fact, we will give the symbols E to p_0 and $-\overrightarrow{\mathbf{p}}$ to the other components. Afterwards, we justify these names, showing that E acts like energy and $\overrightarrow{\mathbf{p}}$ acts like the non-relativistic momentum.

noindent**Definition** *4-momentum* The 4-momentum is a form $\underline{\mathbf{p}}$ defined by

$$p_a \equiv m g_{ab} U^b = m U_a. \tag{3.108}$$

The components of the 4-momentum will be given names:

$$p_a \equiv (E, -\overrightarrow{\mathbf{p}}). \tag{3.109}$$

The raised form of the 4-momentum $p^a \equiv g^{ab} p_b$ is simply

$$p^a = mU^a. \tag{3.110}$$

In Special Relativity where $g_{ab} = \eta_{ab}$ the raised form of the 4-momentum is

$$p^a = \eta^{ab} p_b = \begin{pmatrix} E \\ \vec{\mathbf{p}} \end{pmatrix}. \tag{3.111}$$

This implies

$$p^a = \begin{pmatrix} \gamma m \\ \gamma m \vec{\mathbf{V}} \end{pmatrix} \tag{3.112}$$

so that

$$\vec{\mathbf{p}} = \gamma m \vec{\mathbf{V}}. \tag{3.113}$$

The right hand side gives the non-relativistic 3-momentum, apart from the factor of γ (which is very nearly 1 in non-relativistic situations). Thus we are justified in our choice of the symbol $\vec{\mathbf{p}}$ for the spatial components of the 4-momentum.

The 0'th component of $\underline{\mathbf{p}}$ resembles the non-relativistic energy, plus just a bit extra:

$$\begin{aligned} E = p^0 &= \gamma m & (3.114)\\ &= m\left(1 - V^2\right)^{-1/2} & (3.115)\\ &= m\left(1 + \frac{1}{2}V^2 + O(V^4)\right) & (3.116)\\ &\approx m + \frac{1}{2}mV^2 + O(V^4). & (3.117) \end{aligned}$$

We interpret this as the rest mass (m) + kinetic energy $(mV^2/2)$ + relativistic correction $(O(V^4))$.

In the rest frame $\gamma = 1$ and we have Einstein's famous equation

$$\boxed{E = m}. \tag{3.118}$$

(or in non-relativistic units $E = mc^2$).

Exercise 3.7

a. Show that if the particle has three-velocity $\vec{\mathbf{V}}$, $\vec{\mathbf{p}} = E\vec{\mathbf{V}}$.

b. Show that $E^2 = |\vec{\mathbf{p}}|^2 + m^2$.

3.9.2 Forces

Newton:

Classically: $\overline{\mathbf{F}} = m\overline{\mathbf{a}}$

In Special Relativity, this becomes

$$f_a = m a_a \tag{3.119}$$

where

$$a^a \equiv \frac{\mathrm{d}U^a}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 X^a}{\mathrm{d}\tau^2}. \tag{3.120}$$

Theorem:

$$a_a U^a = 0 \tag{3.121}$$

i.e. the 4-acceleration is perpendicular to the 4-velocity.

Proof:

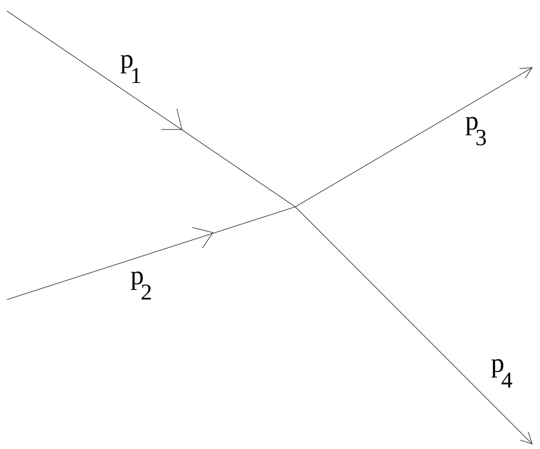
$$\begin{aligned} \overline{a} \cdot \overline{\mathbf{U}} &= \frac{\mathrm{d}\overline{\mathbf{U}}}{\mathrm{d}\tau} \cdot \overline{\mathbf{U}} && (3.122) \\ &= \frac{1}{2} \frac{\mathrm{d}\left(\overline{\mathbf{U}} \cdot \overline{\mathbf{U}}\right)}{\mathrm{d}\tau} && (3.123) \\ &= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\tau}(1) && (3.124) \\ &= 0. && (3.125) \end{aligned}$$

Corollary Since the force $\underline{\mathbf{f}} = m\underline{a}$, with m scalar, we also have

$$\boxed{\underline{\mathbf{f}} \cdot \overline{\mathbf{U}} = 0}. \tag{3.126}$$

3.9.3 Energy-Momentum Conservation

Consider 2 particles colliding:



Total 4-momentum

before	$p_1^a + p_2^a$	(3.127)
after	$p_3^a + p_4^a$	(3.128)

Conservation of Energy and Momentum:

$$\Longrightarrow p_1^a + p_2^a = p_3^a + p_4^a \tag{3.129}$$

3.9.4 Photons

The four-velocity becomes ill-defined when $V \to 1$:

$$U^a = \begin{pmatrix} \gamma \\ \gamma \overrightarrow{\mathbf{V}} \end{pmatrix} \to \begin{pmatrix} \infty \\ \infty \overrightarrow{\mathbf{V}} \end{pmatrix}. \tag{3.130}$$

Fortunately, the 4-momentum still makes sense. Let $m \to 0$ while $V \to 1$, keeping $E = \gamma m$ constant:

$$p_a = (\gamma m, -\gamma m \overrightarrow{\mathbf{V}}) = (E, -E\overrightarrow{\mathbf{V}}). \tag{3.131}$$

Note that $|\underline{\mathbf{p}}|^2 = m^2$.

The 3-vector $\overrightarrow{\mathbf{V}}$ becomes a unit vector as its modulus $V \to 1$. Let $\hat{k} = \lim_{V\to 1} \overrightarrow{\mathbf{V}}$. Then $p_a = (E, -E\hat{k})$. Here $\hat{k}$ tells us the direction of travel of the photon.

In quantum theory, $E = \hbar\omega$ for a photon, where ω is the angular frequency of the light. This implies

$$p_a = \hbar(\omega, -\overrightarrow{\mathbf{k}}) \tag{3.132}$$

where $\overrightarrow{\mathbf{k}} = \omega\hat{k}$. We will write $k_a = (\omega, -\overrightarrow{\mathbf{k}})$ for the wave-number

$$\implies \boxed{\underline{\mathbf{p}} = \hbar\underline{\mathbf{k}}}. \tag{3.133}$$

N.B. The world-line of a photon cannot be parameterized by τ (proper time), since proper time does not exist for a photon ($d\tau = 0$ along the path of a photon). We can still use other parameters, for example the coordinate time t in some reference frame.

Chapter 4

Maxwell's Equations in Tensor Form

'Well, I do not mind telling you I have been at work upon this geometry of Four Dimensions for some time. Some of my results are curious. For instance, here is a portrait of a man at eight years old, another at fifteen, another at seventeen, another at twenty-three, and so on. All these are evidently sections, as it were, Three-Dimensional representations of his Four-Dimensioned being, which is a fixed and unalterable thing.

4.1 Maxwell's Equations – Review

We will use units where $\epsilon_0 = \mu_0 = c = 1$ (Vacuum Equations)

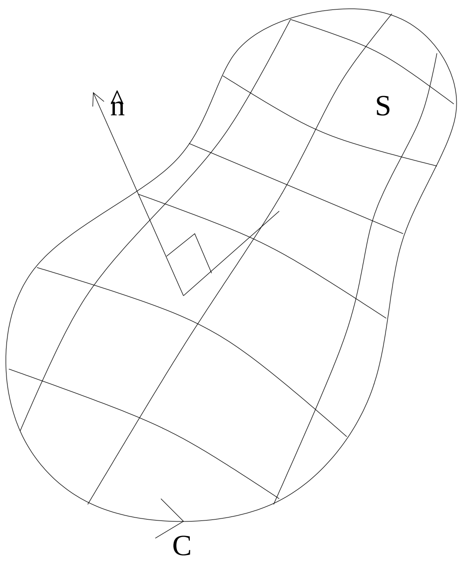
4.1.1 Internal Structure Equations

The internal structure equations involve the fields only; matter terms involving charges and currents do not appear.

$$\nabla \cdot \overrightarrow{\mathbf{B}} = 0, \tag{4.1}$$

$$\nabla \times \overrightarrow{\mathbf{E}} + \partial_t \overrightarrow{\mathbf{B}} = 0. \tag{4.2}$$

Equation (4.1) implies there are no magnetic monopoles - lines of magnetic flux have no endpoints. The meaning of Equation (4.2) can be seen by integrating over a surface S bounded by a curve C:



$$\int_S \left(\nabla \times \vec{\mathbf{E}}\right) \cdot \hat{n} \quad \mathrm{d}^2x = -\int_S \partial_t \vec{\mathbf{B}} \cdot \hat{n} \quad \mathrm{d}^2x \tag{4.3}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \vec{\mathbf{B}} \cdot \hat{n} \quad \mathrm{d}^2x \tag{4.4}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}t} [\text{magnetic flux through } S] \tag{4.5}$$

but, by Stokes' theorem

$$\int_S \left(\nabla \times \vec{\mathbf{E}}\right) \cdot \hat{n} \quad \mathrm{d}^2x = \int_C \vec{\mathbf{E}} \cdot \vec{\mathrm{d}l} \tag{4.6}$$

$$= [\text{electric power round } C] \tag{4.7}$$

Thus, changes in the magnetic flux produce electric power (and vice-versa).

4.1.2 Source Equations

$$\nabla \cdot \vec{\mathbf{E}} = \rho_c, \tag{4.8}$$

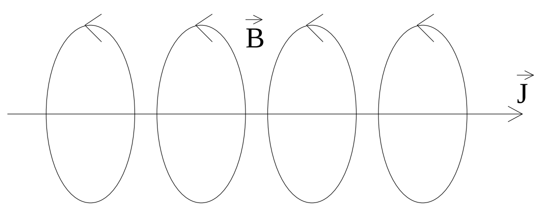
$$\nabla \times \vec{\mathbf{B}} - \partial_t \vec{\mathbf{E}} = \vec{\mathbf{J}}\,. \tag{4.9}$$

Equation (4.8) implies that electric field lines start and stop at electric charges.

For non-relativistic applications, $\partial_t \vec{\mathbf{E}}$ is small, and equation (4.9) gives

$$\nabla \times \vec{\mathbf{B}} \approx \vec{\mathbf{J}} \tag{4.10}$$

i.e. Magnetic field lines circle currents



Maxwell's equations give us 4 vector equations, but 8 component equations.

4.1.3 Lorentz Force

The source equations tell us how matter generates fields. We need a supplemental equation to see how fields affect matter - the Lorentz Force equation. For a particle of charge q

$$\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \vec{\mathbf{V}} \times \vec{\mathbf{B}}\right). \tag{4.11}$$

4.1.4 Charge Conservation

Charge conservation is expressed by the equation

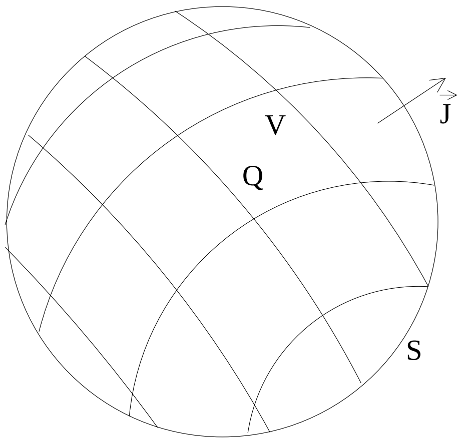
$$\partial_t \rho_c + \nabla \cdot \vec{\mathbf{J}} = 0. \tag{4.12}$$

If we integrate this over a volume V, bounded by the surface S, containing charge Q;

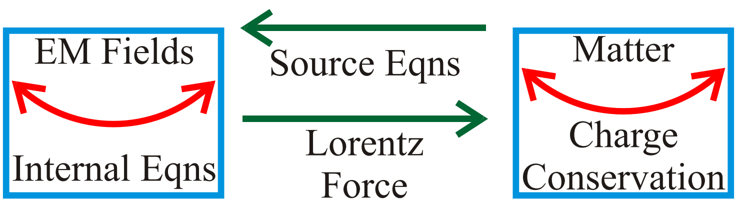
$$\int_V \partial_t \rho_c \, d^3x = -\int_V \nabla \cdot \vec{\mathbf{J}} \, d^3x \tag{4.13}$$

$$\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho_c \, d^3x = -\oint_S \vec{\mathbf{J}} \cdot \hat{n} \, d^2x \qquad \text{(By the Divergence Theorem)} \tag{4.14}$$

$$\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}(Q) = -\,[\text{flow of charge out of } V] \tag{4.15}$$



A physical system of fields and matter can be represented as follows:



4.2 The Faraday Tensor

We define the *Faraday Tensor* as the antisymmetric tensor

$$F_{ab} = \overbrace{\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}}^{b} \Bigg\} a. \tag{4.16}$$

Definition 4.1 *two-forms* A two-form is an antisymmetric second rank tensor with two lower indices.

Thus the Faraday tensor is a two-form. In general, we will define fields as forms (like gradients), and particles (i.e. 4-velocities, currents) as vectors. However, at times we can raise or lower using the metric. e.g. $U_a = g_{ab}U^b$.

Exercise 4.1

a. *Find the raised version of F_{ab}, i.e. find $F^{cd} = \eta^{ce}\eta^{df}F_{ef}$. Be careful if you use matrix multiplication!!*

b. *Next find 'the dual Faraday tensor'*

$$^*F^{ab} \equiv 1/2\, \epsilon^{abcd}F_{cd}. \tag{4.17}$$

Answer:

$$^*F^{ab} = \begin{pmatrix} 0 & +B_x & +B_y & +B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix} \tag{4.18}$$

4.3 Internal Structure Equations

$$\boxed{\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0} \tag{4.19}$$

True for any $a, b, c = 0, 1, 2, 3$.

Example 4.1 $a = 1, b = 2, c = 3$

$$\begin{aligned} \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} &= 0 \\ \implies \frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y + \frac{\partial}{\partial z}B_z &= 0 \\ \therefore \nabla \cdot \overrightarrow{\mathbf{B}} &= 0 \end{aligned}$$

which is the first Maxwell equation.

Note: There are 64 combinations of a, b, c, but *most* are useless!

Example 4.2 $a = 1, b = 2, c = 2$

$$\begin{aligned} \partial_1 F_{22} + \partial_2 F_{21} + \partial_2 F_{12} &= 0 \\ \implies \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(-B_z) + \frac{\partial}{\partial y}(B_z) &= 0 \\ \implies 0 &= 0 \end{aligned}$$

Which is true automatically, and tells us nothing.

Only 4 choices of a, b, c are useful – those where all three are different.

Exercise 4.2 *Consider the equation*

$$\partial_b {}^*F^{ab} = 0. \tag{4.20}$$

Find the four equations for $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$ generated by letting $a = 0$, $a = 1$, $a = 2$, and $a = 3$. Show that these are just the Internal Maxwell equations

$$\nabla \cdot \overline{\mathbf{B}} = 0, \qquad \nabla \times \overline{\mathbf{E}} + \frac{\partial \overline{\mathbf{B}}}{\partial t} = 0. \tag{4.21}$$

4.4 Source Equations

$$\boxed{\partial_b F^{ab} = j^a}. \tag{4.22}$$

where: $j^0 = \rho_c$, $(j^1, j^2, j^3) = \overrightarrow{\mathbf{J}}$, and

$$\begin{aligned} F^{ab} &= \eta^{ac}\eta^{bd}F_{cd}, \qquad \text{(Special Relativity);} &(4.23)\\ F^{ab} &= g^{ac}g^{bd}F_{cd}, \qquad \text{(General Relativity).} &(4.24) \end{aligned}$$

Special Relativity:

$$F^{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{4.25}$$

There are 4 equations, for $a = 0, 1, 2, 3$
E.g. $a = 0$:

$$\begin{gathered} \partial_b F^{0b} = j^0 \\ \partial_0 F^{00} + \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} = j^0 \\ \partial_t(0) + \partial_x E_x + \partial_y E_y + \partial_z E_z = \rho_c \\ \therefore \nabla \cdot \overrightarrow{\mathbf{E}} = \rho_c \end{gathered}$$

4.5 Charge Conservation

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \vec{\mathbf{J}} = 0 \tag{4.26}$$

$$\Rightarrow \partial_0 j^0 + \left(\partial_1 j^1 + \partial_2 j^2 + \partial_3 j^3\right) = 0 \tag{4.27}$$

or

$$\boxed{\partial_a j^a = 0}. \tag{4.28}$$

This equation follows immediately from the Source equation:

$$\partial_a j^a = \partial_a \partial_b F^{ab} = 0 \tag{4.29}$$

as $\partial_a \partial_b$ is antisymmetric while F^{ab} is symmetric.

The 4-divergence

In general, when the *4-divergence*, $\partial_a V^a$, of a vector field vanishes, we say that V^a is conserved.

The 4-current is $j^a = \begin{pmatrix} \rho_c \\ \vec{\mathbf{J}} \end{pmatrix}$. The time component $j^0 = \rho_c$ gives the amount of charge moving in the time direction per unit (space) volume. The components of $\vec{\mathbf{J}}$ give the amount of charge moving in each space direction per unit time.

Exercise 4.3 *Let inertial frame B move at speed $V\hat{x}$ with respect to inertial frame A. Suppose in frame A the magnetic field components vanish. Using the Faraday Tensor, find the magnetic field components and electric field components in frame B.*

4.6 Lorentz Force

$$\boxed{f^a = qU_b F^{ba}}. \tag{4.30}$$

Example 4.3 What is the $a = 1$ component of the force? *Solution*

$$\begin{aligned}
& f^1 = qU_b F^{b1} \\
\Longrightarrow \quad & f^1 = q\gamma\left(F^{01} - V_x F^{11} - V_y F^{21} - V_z F^{31}\right) \qquad (U_a = (\gamma, -\gamma\vec{\mathbf{V}})) \\
& \quad = q\gamma\left(E_x - V_x(0) - V_y(-B_z) - V_z B_y\right) \\
& \quad = q\gamma\left(E_x + (V_y B_z - V_z B_y)\right) \\
& \quad = q\gamma\left(E_x + (\vec{\mathbf{V}} \times \vec{\mathbf{B}})_x\right)
\end{aligned}$$

and so

$$\left(f^1, f^2, f^3\right) = q\gamma\left(\vec{\mathbf{E}} + \nabla \cdot \vec{\mathbf{B}}\right)$$

Checking that $f^\alpha U_\alpha = 0$

$$\begin{aligned}
f^\alpha U_\alpha &= \left(qU_\beta F^{\beta\alpha}\right) U_\alpha \\
&= qF^{\beta\alpha} U_\beta U_\alpha
\end{aligned}$$

but $F^{\beta\alpha}$ is anti-symmetric, while $U_\beta U_\alpha$ is symmetric; the double contraction therefore gives 0.

Exercise 4.4 *Express the Lorentz scalars*

$$F^{ab}F_{ab}, \qquad {}^*F^{ab}F_{ab} \tag{4.31}$$

in terms of $\overline{\mathbf{E}}$ *and* $\overline{\mathbf{B}}$*. Suppose that* $\overline{\mathbf{E}}$ *vanishes in some inertial frame. Show that* $\overline{\mathbf{E}}$ *must be perpendicular to* $\overline{\mathbf{B}}$ *in all frames. Is it possible for* $\overline{\mathbf{B}} = 0, \overline{\mathbf{E}} \neq 0$ *in one frame and* $\overline{\mathbf{E}} = 0, \overline{\mathbf{B}} \neq 0$ *in another?*

4.7 Potential Form

Definition 4.2 *Electromagnetic potential* The electromagnetic potential is given by the form

$$\underline{\phi} = (\phi_e, -\overrightarrow{\mathbf{A}}) \tag{4.32}$$

where ϕ_e is the static electric potential and $\overrightarrow{\mathbf{A}}$ is the magnetic vector potential.

The Faraday tensor involves antisymmetric derivatives of $\underline{\phi}$:

$$\boxed{F_{ab} = \partial_b\phi_a - \partial_a\phi_b}\,. \tag{4.33}$$

This definition is consistent with the assignments

$$\overrightarrow{\mathbf{E}} = -\nabla\phi_e - \partial_t\overrightarrow{\mathbf{A}}, \tag{4.34}$$

$$\overrightarrow{\mathbf{B}} = \nabla\times\overrightarrow{\mathbf{A}}. \tag{4.35}$$

4.7.1 Advantage – Internal Structure Equations

These are automatically satisfied:

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = \,?$$

Using the electromagnetic potential,

$$\implies \partial_a\left(\partial_c\phi_b - \partial_b\phi_c\right) + \partial_b\left(\partial_a\phi_c - \partial_c\phi_a\right) + \partial_c\left(\partial_b\phi_a - \partial_a\phi_b\right) = 0$$

just by cancellations.

4.7.2 Advantage – Source Equations

The equation

$$\partial_b F^{ab} = j^a \tag{4.36}$$

becomes

$$\partial_b\left(\partial^b\phi^a - \partial^a\phi^b\right) = j^a \tag{4.37}$$

where $\phi^a = g^{ab}\phi_b$.

We can write this as

$$\Box^2 \phi^a - \partial^a \left(\partial_b \phi^b\right) = j^a$$

where:

$$\begin{aligned}\Box^2 &\equiv \text{d'Alembertian} \\ &= \partial_b \partial^b \\ &= \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \\ &= \frac{\partial^2}{\partial t^2} - \nabla^2\end{aligned}$$

Thus, Maxwell's equations reduce to a single source equation, with the internal equations being automatic and no longer needed.

4.8 Gauge Transformations

Recall that $\overrightarrow{\mathbf{B}} = \nabla \times \overrightarrow{\mathbf{A}}$. Suppose we apply the *gauge transformation* $\overrightarrow{\mathbf{A}}' = \overrightarrow{\mathbf{A}} + \nabla\psi$ for some function ψ. Then

$$\overrightarrow{\mathbf{B}}' = \nabla \times \overrightarrow{\mathbf{A}}' \quad (4.38)$$

$$= \nabla \times \overrightarrow{\mathbf{A}} + \nabla \times \nabla\psi \quad (4.39)$$

$$= \nabla \times \overrightarrow{\mathbf{A}} + 0 \quad (4.40)$$

$$= \overrightarrow{\mathbf{B}}. \quad (4.41)$$

Similarly, if $\underline{\phi}' = \underline{\phi} + \partial\psi$, then

$$\begin{aligned}F'_{ab} &= \partial_b \left(\phi_a + \partial_a \psi\right) - \partial_a \left(\phi_b + \partial_b \psi\right) \\ &= \partial_b \phi_a - \partial_a \phi_b + \left(\partial_b \partial_a \psi - \partial_a \partial_b \psi\right) \\ &= F_{ab}.\end{aligned}$$

The potentials are therefore not unique, and we are free to choose the most convenient potential, ϕ_a, to solve the problem

4.9 Lorentz Gauge

Suppose we try a potential, $\underline{\phi}'$, where

$$\partial_a \phi'^a = h$$

for some function h. Then we may apply the gauge transformation

$$\underline{\phi}' = \underline{\phi} + \partial\psi$$

where ψ satisfies

$$\partial_a \partial^a \psi = h.$$

This equation can be shown to always have a solution, and so we are left with a new potential $\underline{\phi}$ which satisfies

$$\boxed{\partial_a \phi^a = 0}. \qquad \text{"Lorentz Gauge"} \tag{4.42}$$

In Lorentz gauge, the source equation becomes

$$\boxed{\Box^2 \phi^a = j^a}. \tag{4.43}$$

4.10 Light Waves

In the vacuum, $j^a = 0$

$$\implies \Box^2 \phi_a = 0 \tag{4.44}$$

$$\implies \partial_b \partial^b \phi^a = 0 \tag{4.45}$$

$$\implies \frac{\partial \phi^a}{\partial t^2} - \frac{\partial \phi^a}{\partial x^2} - \frac{\partial \phi^a}{\partial y^2} - \frac{\partial \phi^a}{\partial z^2} = 0 \tag{4.46}$$

which is the wave equation, with solution of the form

$$\phi^a = C^a e^{ik_b x^b} \tag{4.47}$$

$$= C^a e^{i(\omega t - \overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{x}})}. \tag{4.48}$$

Here $\underline{\mathbf{k}} = (\omega, -\overrightarrow{\mathbf{k}})$ is the wave vector, and C^a is the amplitude.

Check:

$$\frac{\partial^2 \phi^a}{\partial t^2} = -\omega^2 \phi^a \tag{4.49}$$

$$\frac{\partial^2 \phi^a}{\partial x^2} = -k_x^2 \phi^a \tag{4.50}$$

$$etc \ldots \tag{4.51}$$

$$\tag{4.52}$$

Substituting these into the wave equation gives

$$-\omega^2 + \overrightarrow{\mathbf{k}}^2 = 0 \tag{4.53}$$

or

$$\omega = \pm \left| \overrightarrow{\mathbf{k}} \right|. \tag{4.54}$$

Exercise 4.5 *Suppose that magnetic monopoles exist in nature. Then, in addition to the electric charge-current 4-vector* $\bar{\mathbf{j}}_e$*, there is a magnetic charge-current 4-vector* $\bar{\mathbf{j}}_m = (\rho_m, j_{m\,x}, j_{m\,y}, j_{m\,z})$ *where* ρ_m *is the magnetic charge density and* $j_{m\,x}$ *is the current of magnetic charge in the* x *direction. The Maxwell equations become*

$$\partial_b F^{ab} = j_e^a$$
$$\partial_b {}^*F^{ab} = j_m^a.$$

a. Consider the second equation $\partial_b {}^*F^{ab} = j^a_m$*. Find the four equations for* $\overrightarrow{\mathbf{E}}$ *and* $\overrightarrow{\mathbf{B}}$ *generated by letting* $a = 0$*,* $a = 1$*,* $a = 2$*, and* $a = 3$*.*

b. Show that magnetic charge is conserved; i.e. show that

$$\partial_a j^a_m = 0.$$

c. The Lorentz force on a magnetic monopole of charge q_m *and 4-velocity* U^a *is*

$$f^a = \frac{\mathrm{d}p^a}{\mathrm{d}\tau} = q_m U_b {}^*F^{ab}.$$

Find the four equations generated by letting $a = 0$*,* $a = 1$*,* $a = 2$*, and* $a = 3$*. Express these in terms of the three-velocity* $\overrightarrow{\mathbf{V}} = \mathrm{d}\overrightarrow{\mathbf{x}}/dt$ *and* $\gamma = (1 - V^2)^{-1/2}$*.*

d. Show that the Lorentz force in the previous item is perpendicular to $\overline{\mathbf{U}}$ *in the sense that*

$$\underline{\mathbf{f}} \cdot \overline{\mathbf{U}} = 0.$$

e. Suppose that the Faraday tensor F_{ab} *can be written in the form*

$$F_{ab} = \partial_a \phi_b - \partial_b \phi_a$$

for some four-potential $\underline{\phi}$*. Show that the magnetic current 4-vector must vanish, i.e.* $\bar{\mathbf{j}}_m = 0$*.*